Lecture Notes on Computational Macroeconomics

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09/04/98, A.M.

Subject: Introduction
Reference: course syllabus

Basically, in this course we are going to study three modules. In module I, we deal with issues of monetary economics in models without capital; in module II, we deal with a series of extended neoclassical growth model in which no money but capital is involved; in module III, we deal with models with both money and capital.

Module I: Monetary economics without capital

Week 1: introduction

The model here has no money and no capital, which is the easiest one. We also explain how we introduce money into the model. The Blanchard (1990) paper provides a good survey at least for Module I.

Week 2: nominal indeterminacy

This is a topic Canzoneri, Diba and Cumby are currently working on. This is a direct application of monetary model without capital. Using the cumulative monetary stock, we can raise the issue of nominal determinacy, or the uniqueness of price level. After they set up a monetary model where price level is not unique, they will then call on a fiscal model so that even if the price level is not unique in the monetary regime, the proper fiscal policy would make the price level unique.

Week 3: neutrality, super-neutrality and welfare costs

We are talking about some classical topics in monetary economics here. Monetary neutrality means the change in the stock of money won't have any real effect in the economy, whereas monetary super-neutrality means the change in the growth rate of money won't affect the real economy. Welfare costs are related to the change in the growth rate of money, and that is why those three topics are jointly talked about here.

Week 4: nominal rigidities via predetermined prices
Week 5: nominal rigidities via fixed prices

These days when we talk about two-period contract, we mainly mean the Taylor one, not the Fischer one. The Fischer contract concerns with predetermined prices whereas the Taylor type of contracts concern with fixed price. When we are saying predetermined price we mean that in **each** period, the price level is a function of the price level in the exactly previous period. Suppose an agent can always make two-period decision of wage levels, then he/she constructs two fixed price levels. Suppose there are two agents in our model each period, then the fixed price levels are staggered for every two periods with certain overlap. Although the Taylor (1979) paper is shorter than the Taylor (1980) paper, it contains more essence than the longer one.

Module II: Capital accumulation dynamics without money

Week 6: why do we study module II in monetary economics

This is pretty much the same stuff we have done with Diba. We can say this is real business cycle model, or we can equally well interpret it as a neoclassical growth model. Both concepts are essentially the same. The confusion of the two names comes from the fact that neoclassical growth model uses continuous time and deterministic setting whereas the business cycle scholars use discrete time and stochastic setting.

Week 7: monopolistic competition and increasing returns to scale

In the models we studied with Diba, the market was assumed to be perfect competitive and we use constant returns to scale. However, in our model here, producers, or households, are setting price levels, which is not supported by the assumption of perfect competition. To make it possible for households to set prices, we need to introduce market power. If we allow monopoly to exist, then increasing returns to scale will appear.

Week 8: real indeterminacy

On week 2, we discussed nominal indeterminacy. Here we are talking about the uniqueness of real level, which comes from the possibility of increasing constant scale.

Week 9: investment adjustment cost

The standard model of capital accumulation is $K_{t+1} = (1-\delta)K_t + I_t$. However, this model doesn't treat well with monetary effect. There are only three possible values for investment in continuous time, namely 0, $-\infty$ or $+\infty$, which is not realistic. This is the real side reason why we study investment adjustment cost, instead of the standard model. The nominal side reason is that the standard model cannot produce some monetary effect. The study on investment adjustment costs will provide some more plausible monetary effect.

Module III: Models with both money and capital

Week 10: why do we need to use the computer

Finally we are going to confront the intimidating models with both money and capital. Once we add capital stock in monetary models, we should have dynamic decisions, which will make our life much difficult. The reason why we need computer as an aid here is because we cannot solve higher order equations.

Week 11: a simple model with an analytical solution

The reason why we can solve this model is that the fourth order equation can be factored out as the product of two second order equations which are both solvable and economically meaningful.

09/04/98, P.M.

Subject: Consumption, bond and money
Reference: Notes on consumption, bond and money, without capital

The simplest consumption model is the so-called Lucas' tree model, where the output level Y is exogenously given. The typical form of the model is:

[A] $\max U(C)$, s.t. C = Y.

If goods are perishable, then we don't have the choice of saving that will smooth income. However, once we introduce the possibility of saving, then the optimal consumption plan depends upon both past and future income. Similarly, in a setting of Capital Asset Pricing Model, we can also interpret it as the fact that the current asset prices depend upon both past and future prices. If we specify the saving in the way that saving rate is exogenously given, then we get the extension to Solow-Swan model:

[B] $\max U(C)$, s.t. $S_t = \Delta Y_t$.

If we endogenize the saving rate, then we can get the extension to Ramsey-Cass-Koopman model.

So far, we haven't consider the role of labor yet. If we get rid of the assumption of exogenous output, and introduce labor into the utility function, then we can get the following model:

[C] $\max U(C, L)$, s.t. $C = AL^{1-\alpha}$.

This can be interpreted as following: the more we work, the more can we consume, but less leisure will we consume.

Moreover, we can consider time preference in the consumption model, which is represented by the intertemporal trade model. The simplest one of this category is the two-period model:

[D] $\max U(C_1, C_2)$, s.t. $C_1 + C_2 / R = Y_1 + Y_2 / R$.

Of course we can extend this model further to multi-period model, even infinite horizon model. But so far, our model is deterministic. If we introduce uncertainty into the model, then the goal of our representative household is to maximize the **expected** utility level. According to the criteria of discrete/continuous and deterministic/stochastic, we can have the following four categories of models:

	Discrete	Continuous
Deterministic	Model [D]	Growth Model
Stochastic	Business Cycle Model	Ito's Lemma/Wiener Process

Now, let's consider a multi-period model. One of the constraints that depicts the rule of accumulating wealth is $A_{i+1} = R_{i+1}(A_i + Y_i - C_i)$, $i = t, ..., T-1$. Suppose the initial wealth level is zero, i.e., $A_t = 0$. Considering the possibility of bequeath motive, we have the terminal condition of $A_{T+1} \geq 0$. To simplify the model, we can make the utility function additively separable across time periods. Then we have the following objective function:

[E] $\max \sum_{i=t}^{T} \beta^{i-t} V(C_i)$, where β is the constant discount rate.

We can solve this model in three ways: First, use Kuhn-Tucker conditions to solve the constrained optimization problem. Second, for the case of finite-period recursive model, we can also solve the model using the backward deduction method. Third, for the case of infinite-horizon model, we can use the method of Belma Equation or Value Function [????], i.e., to transform all the budget constraints into one.

The last thing we are going to do here today is to introduce money. How can we introduce money into the budget constraint? Considering the original budget constraint in model [E], we can split the wealth into two components, interest-bearing bonds and non-interest-bearing money, thus we get the following budget constraint: $B_{i+1} + M_{i+1} = R_i(B_i + Y_i - C_i) + M_i$.

If we don't introduce money into the utility function, then it is apparent that the optimal thing for the agent to do is to hold zero amount of money. There are basically three ways to make it optimal for the representative agent to hold money in the model as in the real world: First, by Cash-In-Advance constraints, i.e., $C_i \le M_i / P_i$ or $C_i \le M_{i-1} / P_i$. Second, put money into the utility function directly, as the Sidrauski model does. Essentially the MIU model $U(C_i, M_i / P_i)$ can yield the CIA model as a special case by using Leontief technology, i.e., by not allowing the substitution between consumption and holding money balance. Then we have to figure out a way to interpret the value of holding money balance. Third, by introducing the concept of transaction cost. Let $C_i \cdot f(M_i / P_i)$ be constant, and $f'(\cdot) < 0$. To save the same amount, which is the constant, if we carry more cash, then we consume more. This explains the motive of carrying cash very well.

Of course, we have other places to introduce money as well, using the similar idea. Say, the more we carry cash, the less we lose out of labor income or endowment. Then we can introduce labor into the utility and use the concept of labor cost to interpret the model. If we carry more cash, then we don't go to bank often, and thus we have more time to work.

Example:

$$\max_{C_1, C_2} \frac{C_1^{1-\gamma} - 1}{1-\gamma} + \beta \frac{C_2^{1-\gamma} - 1}{1-\gamma} \text{ s.t. } C_1 + \frac{C_2}{R} \le Y_1 + \frac{Y_2}{R}.$$

The solutions to this problem are: $C_1 = \dfrac{R^{\frac{1-\gamma}{-\gamma}}}{R^{\frac{1-\gamma}{-\gamma}} + \beta^{\frac{1}{\gamma}}} \left(Y_1 + \dfrac{Y_2}{R} \right)$, $C_2 = \beta R C_1$.

As far as the discount factor is positive, the fraction in front of the total wealth is between zero and unity. That is to say, for each period, the agent consumes a fraction of the total wealth. The interest stream coming from the total wealth is called permanent income. We have a couple of insights from this solution:

(1) The consumption is positive since the discount factor is positive;

(2) Through the storage technology of holding bonds, the current consumption is affected by the income in the next period, and the next period consumption is also affected by the current income. This is totally different from the endowment economy. Not only the past matters, but the future matters as well. This is also called life cycle income hypothesis or permanent income hypothesis. What matters is the life cycle, which includes the future.

(3) This is a partial equilibrium in the sense that the interest rate R is determined outside. To make the model complete, we could add ex-post market clearing conditions $C_1 = Y_1$ and $C_2 = Y_2$. Combine those ex-post conditions with the ex-ante optimizing conditions, namely, replacing C_1 with Y_1 in the solution for C_1, we get the general equilibrium interest rate level $R = \beta^{-1} Y_2 / Y_1$. If $Y_2 = Y_1$, then $R = \beta^{-1}$, i.e., the market interest rate is equal to the agent's discount rate. If $Y_2 > Y_1$, then $R > \beta^{-1}$, i.e., the market interest rate is greater than the discount rate so that it is attractive to save today.

Alternatively, if we want to extend the Lucas' tree model into a general equilibrium model, then ex-ante we have $C_t = Y_t$, and ex-post we impose $C_1 + \frac{C_2}{R} \le Y_1 + \frac{Y_2}{R}$ so that R is determined.

09/11/98, A.M.& P.M.

Subject: Nominal Indeterminacy
Reference: Kim, Jinill "A primer on price level determinacy: monetary policy in discrete time and fiscal policy in continuous time"
Benhabib, Schmitt-Grohe, and Uribe (1998) "Monetary policy and mutiple equilibria" mimeo

We've discussed the simplest endowment economy, i.e., $\max U\left(\{C_i\}_{i=0}^{\infty}\right)$ s.t. $C_i = Y_i$. One purpose of this model is the Asset Pricing Model in Finance. Their purpose is to use a specific utility function to interpret the asset pricing mechanism. Today, we are going to use this endowment economy to discuss price indeterminacy. Note that we are using a deterministic setting there.

If we want to explain price indeterminacy, at least we should put money into the endowment economy, then we get some kind of IS-LM model. Or, if we don't put money in the model, we can put bonds into it. Of course we can put both in the model.

We first go over a simple discrete-time model, where the fiscal policy is very simple, to understand what the price level indeterminacy is. Then we do a continuous-time model to investigate the interaction between monetary policy and fiscal policy.

1. Monetary Policy in Discrete Time

The representative agent's optimization problem is

$$\max_{\{C_t, M_t, B_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U\left(C_t, M_t / P_t\right) \tag{1}$$

$$\text{s.t. } C_t + T_t + \frac{M_t + B_t}{P_t} = Y_t + \frac{\mathcal{M}_t M_{t-1} + I_{t-1} B_{t-1}}{P_t} \tag{2}$$

We didn't specify the utility function form yet, but we can refer to the paper by Benhabib, Schmitt-Grohe and Uribe (1998), which provides a good survey on different utility function forms. Regarding the budget constraint, we spent a little bit more time. In the intermediate macro, our regular budget constraint is $C + S + T = C + G + I$, whereas we discard investment here. On the aspect of spending for the consumer, the LHS of the budget constraint consists of consumption, tax by the government, and new holding of money and bonds. The available real goods for the consumer, in the RHS of the budget constraint, consists of endowment Y_t, and goods can be bought using the stock of money M_{t-1} and bonds B_{t-1} accumulated by the end of last period. Note that we are using the current price level to determine the purchasing power of last period nominal wealth. Since bonds bear interest, we need to add the interest growth part in front of bonds holding. We can use the current period interest growth I_t, which means that in a stochastic stetting we know the distribution of the next period interest rate when we buy bonds. But in a deterministic setting, we don't know I_t when we buy bonds, thus we are using I_{t-1} here. (Suppose we use in this model here, then the perfect foresight has been assumed.)

In most cases, real money balances won't grow by itself so that $\mathcal{M}_t$ should be one. But in Lucas (1972) paper, where he explained the money neutrality issues, he used $\mathcal{M}_t \neq 1$ to represent the idea that people will get some gross return on holing real money balances due to some outside shocks, say helicopter money. He also assumed that people know the distribution of such outside shocks ex-ante, or say people know how often the helicopter will come and how the extra money will be distributed among people. In our stochastic setting later on, what matters is the expected gross growth of money holding.

Set up the Lagrangian in the following way:

$$L = \sum_{t=0}^{\infty} \beta^t \left\{ U(C_t, M_t / P_t) + \lambda_t \left[Y_t + \frac{\mathcal{M}_t M_{t-1} + I_{t-1} B_{t-1}}{P_t} - C_t - T_t - \frac{M_t + B_t}{P_t} \right] \right\}$$

The first-order-conditions are:

$$L_{C_t} = U_{1t} - \lambda_t = 0 \tag{3}$$

$$L_{M_t} = \frac{U_{1t}}{P_t} + \beta \frac{\lambda_{tt+1} \mathcal{M}_{t+1}}{P_{t+1}} - \frac{\lambda_t}{P_t} = 0 \tag{4}$$

$$L_{B_t} = \beta \frac{\lambda_{t+1} I_t}{P_{t+1}} - \frac{\lambda_t}{P_t} = 0 \tag{5}$$

From these FOCs, we can get an inter-temporal condition (also called Euler equation)

$$\frac{U_{1t}}{U_{1,t+1}} = \frac{\beta I_t}{P_{t+1} / P_t}, \tag{6}$$

and an intra-temporal condition (also called liquidity preference)

$$\frac{U_{2t}}{U_{1t}} = 1 - \frac{\mathcal{M}_{t+1}}{I_t}. \tag{7}$$

Essentially the intertemporal condition is the IS curve, whereas the intra-temporal condition is the LM curve. (7) depicts the marginal rate of substitution between real balances holding and consumption. When outside money shocks are bigger, real balances is less desirable to hold thus the consumer would like to use less consumption to exchange for the same amount of real balances. When the interest rate is higher, it is more attractive to hold bonds, thus current consumption is less desirable, and the consumer would like to use more consumption to exchange for the same amount of real balances.

Recall that we use CRRA or power utility function $U(C_t) = \frac{C_t^{1-\gamma} - 1}{1-\gamma}$ before. Since we have one more argument M_t / P_t here, we could use CES function to depict the substitution between consumption and real balances. Hence we specify the utility function as

$$U\left(C_t, \frac{M_t}{P_t}\right) = \frac{\left[\alpha C_t^{1-\sigma} + (1-\alpha)\left(\nu \frac{M_t}{P_t}\right)^{1-\sigma}\right]^{\frac{1-\gamma}{1-\sigma}} - 1}{1-\gamma} \tag{8}$$

The marginal utilities of this utility function are:

$$U_{1t} = \left[\alpha C_t^{1-\sigma} + (1-\alpha)\left(\nu \frac{M_t}{P_t}\right)^{1-\sigma}\right]^{\frac{\sigma-\gamma}{1-\sigma}} C_t^{-\sigma} \alpha, \tag{9}$$

$$U_{2t} = \left[\alpha C_t^{1-\sigma} + (1-\alpha)\left(\nu \frac{M_t}{P_t} \right)^{1-\sigma} \right]^{\frac{\sigma-\gamma}{1-\sigma}} \left(\frac{M_t}{P_t} \right)^{-\sigma} \nu^{1-\sigma}(1-\alpha). \tag{10}$$

When we use the CES function, we know we could have two extreme cases:

Case 1: $\sigma = 1$, then we have the Cobb-Douglas case between consumption and real balances. Furthermore, if $\gamma = 1$, then the two inter and intra-temporal equations become:

$$\frac{1}{C_t} = \frac{1}{C_{t+1}} \frac{\beta I_t}{(P_{t+1}/P_t)}, \tag{11}$$

$$\frac{M_t}{P_t} = \frac{1-\alpha}{\alpha}\left(1 - \frac{\mathcal{M}_{t-1}}{I_t} \right)^{-1} C_t. \tag{12}$$

Case 2: $\sigma = \infty$, then we have the Leontief case between consumption and real balances. The two inter and intra-temporal equations become:

$$\frac{M_t}{P_t} = \frac{1}{\nu} C_t, \tag{13}$$

$$\frac{1}{C_t^{\gamma}} = \frac{1}{C_{t+1}^{\gamma}} \frac{\beta I_t}{(P_{t+1}/P_t)}. \tag{14}$$

The government budget constraint is:

$$T_t + \frac{M_t + B_t}{P_t} = \frac{\mathcal{M}_t M_{t-1} + I_{t-1} B_{t-1}}{P_t}, \tag{15}$$

$$\text{where } \mathcal{M}_t = \left(\frac{M_t}{M_{t-1}} \right)^{\rho}. \tag{16}$$

Regarding the multiplicative shock rules, we may consider two extreme cases. When $\rho = 0$, we have $\mathcal{M}_t = 1$, regardless of the money growth rate; when $\rho = 1$, the shocks rate μ_t is the same as the money growth rate g_t.

So far we have five equations, (2), (13), (14), (15), (16), and seven variables, C, B, M, T, P, I, $\mathcal{M}$. We need two more equations to complete the model, and we can use both fiscal rule and monetary rule to fulfil this need.

Here we assume that fiscal policy follows a balanced-budget rule:

$$\frac{M_t + B_t}{P_t} = \frac{M_{t-1} + B_{t-1}}{P_t}. \tag{17}$$

Essentially the fiscal rules does nothing but change the combination of money and bonds, without changing the sum.

We have seen monetary instruments like $I_t = I^*$ and $M_t = M^*$. But nowadays Fed pays more attention to interest rate. Here we assume that the monetary policy uses interest rate as an instrument and targets money stock and the price levels:

$$I_t = k(M_t)^{\tau_m}(P_t)^{\tau_p}(F_t)^{\tau}, \qquad (18)$$

where the constant k makes possible the existence of the steady state.

To interpret (18), look at $I_t = k(M_t)^{\tau_m}$ at first. When $\tau_m = 0$, we have $I_t = I^*$, i.e., Fed doesn't pay any attention to money supply; when $\tau_m = \infty$, $I_t^{1/\tau_m} = M_t$ implies $M_t = M^*$, i.e., Fed doesn't pay any attention to interest rate. Similarly we can interpret the attention paid to the price level P_t and the funds rate F_t.

The real side of the economy is trivial:

$$C_t = Y_t. \qquad (19)$$

For the sake of simplicity, we assume that $\sigma = \infty$. then the nominal equilibrium is expressed by the following three equations:

$$m_t - p_t = y_t \qquad (20)$$

$$i_t - (p_{t+1} - p_t) = \gamma(y_{t+1} - y_t) \qquad (21)$$

$$i_t - \tau_m m_t - \tau_p p_t = \tau f_t, \qquad (22)$$

where lower-case letters represent the logarithm of upper cases and the constants are neglected due to our interest in dynamics. Combining the three equations, we produce an equation for nominal equilibrium as follows:

$$p_{t+1} = (1 + \tau_m + \tau_p)p_t + \tau f_t + (\gamma + \tau_m)y_t - \gamma y_{t+1}. \qquad (23)$$

Algebra on difference equation shows that a sufficient condition for uniqueness of the price level is: $\tau_m + \tau_p > 0$. (24)

Finally B_t and T_t are determined by the fiscal policy and the government budget constraint.

The condition (24) says that if there is a pressure on the movement of interest rate, Fed should pay attention to either money or price level, according to our reaction rule (18). For example, when we see a price increase, Fed should increase the funds rate.

To understand the meaning of price level indeterminacy, let's consider the following cases:
Case 1: $\tau_m + \tau_p = 0.1$, we have a simplified form $p_{t+1} = 1.1p_t + f_t$;
Case 2: $\tau_m + \tau_p = -0.1$ we have a simplified form $p_{t+1} = 0.9p_t + f_t$.
Let $p_0 = 0$, and there is a one-time shock occurred at period 1 only, namely $f_1 = 0.25$, $f_2 = 0$, $f_3 = 0, \ldots$.

In case 1, since the shock last for only one period, the future price level should go back to the original level without the shock, i.e., $p_2 = 0$. Then $p_2 = 1.1p_1 + f_1$ yields $p_1 = -0.25/1.1$, $p_2 = p_3 = \ldots = 0$. We claim that this is the only set of prices that make economic sense. Otherwise, suppose $p_1 = 0.25$ then all future prices will explode, which is not consistent with the convergence of price level.

In case 2, if we let p_2 immediately go back to the original level that prevails when no shocks occurred, i.e., $p_2 = 0$ then $p_2 = 0.9p_1 + f_1$ yields $p_1 = -0.25/0.9$, $p_2 = p_3 = ... = 0$. However, this is not the only set of prices that make economic sense. Suppose $p_1 = 0$, instead, then $p_2 = 0.25$, $p_3 = 0.9 * 0.25$, $p_4 = 0.9 * (0.25)^2 \ldots$ i.e., the future price level will shrink gradually toward zero, which makes sense. Essentially we can put any number for in this case, or say, we have a problem of price indeterminacy.