Information Economics

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Preface

Just a few sessions into the "Economic Uncertainty and Information" class taught by Professor Anjan V. Thakor in Winter 2001, I realized that an electronic version of the notes, instead of my poor hand writing, would be a valuable resource to keep, update, and refer back to. So I organized the notes electronically while taking the class in that semester. Subsequently I made quite extensive changes to the notes in Winter 2002, when I sat in the same class again in fond of Professor Thakor's energetic teaching and the thought-provoking issues being discussed in the class. Below is a brief overview of the material covered.

This is a class concerning the studies in presence of asymmetric information. What is "asymmetric information" anyway? This is a concept opposed to "full information," a setting of which is usually considered to possess the following characteristics: (1) there is only one single price in the market; (2) the market is cleared; (3) all individuals are price takers; and (4) there is a linear pricing rule, i.e., the price charged is independent of transaction quantities. In a typical game with "asymmetric information", the agent that has more information than the other will use the information advantage strategically to his own good. We often observe the following characteristics in an "asymmetric information" setting: (1) there may not be a single price; (2) firms don't always act as price taker, or "market power" (due to information advantages) does exist (i.e., information produces rents); (3) price charged may depend on quantities sold (or non-linear pricing rule); (4) a competitive equilibrium may not exist.

Don't confuse asymmetric information models with irrational expectation models. In fact, when we mention "rational expectation" models, we meant that the following two conditions are satisfied: (1) there is unbiased expectation of the future among all players, i.e., the priors are "correct;" and (2) all players use Bayes rule to update beliefs. It turns out that the second qualification for "rational expectation" is not as restrictive as we might thought, but the first qualification is really restrictive. Although many people assume that investors have rational expectation, we find that upon evaluating the investors behavior ex post, the second qualification was more or less fulfilled but the empirical data exhibits systematic bias in terms of the first qualification.

Under asymmetric information, prices often convey information. Stiglitz (1971) studies the commodity market for corn. Some people know the average size of corn production ahead of all others, so these privileged folks can trade successfully on the futures market. Once people recognize the superior information possessed by the privileged group, the movement of which on the futures market can be used to infer the private information and thus strategic actions can take place on the spot market as well. Stiglitz's drastic finding was that prices cannot convey too much information, otherwise the market will fail. This is somewhat counter-intuitive because we usually think that market will fail only due to lack of information. However, Stiglitz argues that at the moment when the analysts try to trade using the information they acquired with certain cost, the commodity prices will incorporate such new information immediately so that the information collectors don't even have a chance to trade. Therefore, when the prices convey too much information or too fast, there is no incentive for people to collect information in the first place. Grossman and Stiglitz (1976) further indicate that the strong-form efficiency[1] is simply not even theoretically efficient, a conclusion now known as "Grossman- Stiglitz Paradox."

In the first four chapters, we discuss static models with information asymmetry. In Chapter 1, we cover the famous "lemon's problem", also known as adverse selection problem, and Spence's model in which workers reveal their types to employers by choosing appropriate education level. In Chapter 2, we introduce self-selection models, such as Rothschild-Stiglitz model concerning the insurance market and Ofer-Thakor model regarding firm's decision of cash repayment. We devote Chapter 3 to basic game theoretic knowledge that will be useful for the rest of topics covered in this course. In particular, we carefully explain different refinements of Nash Equilibrium using extensive examples and real world applications. Chapter 4 touches upon principal-agent model.

Starting Chapter 5, we cover some real-world applications related to in-

[1] "Weak-form efficiency" says that past price information cannot predict future prices. "Semi-strong-form efficiency" indicates that all publicly available information cannot predict future prices. "Strong-form efficiency" is achieved when no existing information, public or private, can predict future prices.

formation economics. Reputations model is discussed in Chapter 5. Market microstructure issues and security design are explained in Chapter 7 and 8, respectively. In chapter 6 and 9, we discuss models of herding behavior and behavioral irrationality such as over-confidence and cascades.

Contents

List of Figures

List of Tables

Chapter 1

Market Failure and Signaling

As we know, Akerlof, Spence and Stiglitz won the Nobel Prize in Economics in year 2001 "for their analyses of markets with asymmetric information." So it is not surprising that this class starts with the two Nobel Prize winning papers by the first two laureates.

1.1 Akerlof (1970)

In the real world, we observe many markets in which prices for used goods are "too low." While it is easy to attribute the phenomenon to irrational behavior of market participants, it proves difficult to explain on rational ground. So the research question Akerlof asks in his paper is: why do markets in presence of qualitative uncertainty exhibit low equilibrium prices? A brief answer from Akerlof is that sellers are better informed about the quality and they behave strategically in quality choices, and that the interaction between the existence of asymmetric information and the buyers with rational expectations leads to market failure.

Here is one variation of the Akerlof (1970) paper. There are two goods in the economy that differ in characteristics of quality. The numeraire good 1 has a fixed quality and price, both of which are normalized to be 1. Good 2 has a variable quality, uniformly distributed between 0 and 2, and its price p will be determined endogenously. There are two types of agents in this economy and each agent is considered atomistic in its type. Each type A agent is endowed with only N units of numeraire good 1 and each type B agent is endowed with only M units of good 2. The information asymmetry lies in the fact that only type B agents know the true quality of good 2. Type B agents will potentially sell the quality-varying good 2 to type A agents in exchange of good 1, and both types of agents derive utility from consuming two types of goods.

Both the quantity and quality of goods consumed determine the objective utility level. The objective marginal utility $\mu(s)$ from consuming the s^{th} unit of good 2 diminishes at a constant rate and is equal to the quality at each infinitesimal amount. For example, type B agent enjoys utility 2 from the first $\varepsilon > 0$ unit of good 2, and derives zero utility from the last unit. In the quality-quantity space below, we have $\mu(s) = 2 - ({}^2\!/_N) \cdot s$. When type B agent provides t units of good 2 for trade, it is in his interest and thus common knowledge that he will sell first the good with lowest quality, and the quality of good 2 in trade ranges from 0 to $\mu(M - t) = {}^{2t}\!/_M$.

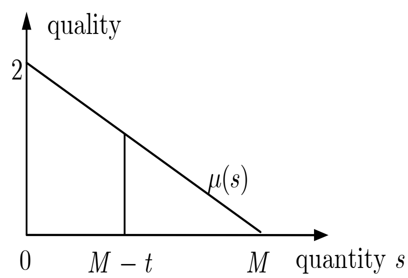

Figure 1.1: Quality-Quantity Pair in an Adverse Selection Model

Due to difference in endowment, two types of agents assign different contribution factors, $c_1 = {}^3\!/_2$ and $c_2 = 1$, to the objective utility level for good 2. Hence the type B agent derives total utility $1 \cdot [{}^1\!/_2 \cdot (2 + {}^{2t}\!/_M) \cdot (M - t)] = M - {}^{t^2}\!/_M$ from consuming $M - t$ units of good 2, and the type A agent derives expected total utility ${}^3\!/_2 \cdot \overline{\mu} \cdot t$ from consuming t units of good 2, where $\overline{\mu}$ is the expected quality of good 2 in trade.

Since the numeraire good has quality fixed at level 1, the derived utility from consuming good 1 is the same as the consumption amount. In order to avoid the confounding effects from the concavity, we assume that the total utility for each agent is simply the sum of the perceived utility from two goods. Denote by x_a^1 and x_b^1 the consumption of good 1 by two types of agents.

Therefore, on the supply side of good 2, a typical type B agent is optimizing

$$\max_t \ x_b^1 + (M - \tfrac{t^2}{M}) \ \text{ s.t. } x_b^1 = p \cdot t,$$

or,

$$\max_t \ p \cdot t + (M - \tfrac{t^2}{M}).$$

The first-order condition to this optimization boils down to $p - {}^{2t}/_{M} = 0$. Hence the supply schedule for good 2 is

$$t = \begin{cases} \frac{1}{2}M \cdot p & \text{if } p \leq 2; \\ M & \text{if } p > 2. \end{cases}$$

On the demand side for good 2, a typical type A agent is optimizing

$$\max_t \ x_a^1 + \tfrac{3}{2}\overline{\mu}t \text{ s.t. } N - x_a^1 = p \cdot t,$$

or,

$$\max_t \ N - pt + \tfrac{3}{2}\overline{\mu}t.$$

The first-order condition to this optimization is $-p + {}^{3}/_{2}\overline{\mu} = 0$. The demand schedule for good 2 is

$$t = \begin{cases} \frac{N}{p} & \text{if } p < \frac{3}{2}\overline{\mu}, & \text{i.e., goods 2 is strictly preferred;} \\ [0, \frac{N}{p}] & \text{if } p = \frac{3}{2}\overline{\mu}, & \text{i.e., indifferent between two goods;} \\ 0 & \text{if } p > \frac{3}{2}\overline{\mu}, & \text{i.e., goods 1 is strictly preferred.} \end{cases}$$

In observance of price p ($p \leq 2$) (since the ratio of marginal utility to price for the numeraire good is 1, the marginal utility and thus its quality should be in line with the price for good 2), buyers know that the supply of goods 2 is $\frac{1}{2}Mp$ and the highest quality available will be $2 - 2 \cdot (M - \frac{1}{2}Mp)/M = p$. The lowest quality will be 0 and thus the rational expectation of the buyers on the quality levels is $\overline{\mu} = \frac{1}{2}p$, which falls into the region $p > \frac{3}{2}\overline{\mu}$. Hence the relevant portion of the demand schedule is effectively zero, i.e., no trade will occur. We usually call this case as the "broken-down lemon market."

Suppose that we use c_2 in this calculation instead of the specific value $\frac{3}{2}$, what is the admissible region of c_2 that allows a "broken-down lemon market?" Everything remains the same except that we need to use c_2 to replace $\frac{3}{2}$ in the demand correspondence. So the region is $c_2 \in (0, 2)$.

1.2 Spence(1974)

While Akerlof observes market breakdown as a very serious consequence of asymmetric information, we don't observe many market failures in the reality despite the fact of asymmetric information. The research question Spence asks is: can we construct a model in which Akerlofian market breakdown can be avoided even when there is asymmetric information? The brief answer here is positive. In a setting of signaling with attribute-specific costs, employees know their own productivity better than the employers, and they can effectively signal their superior information so as to reach an equilibrium.

Employers try to hire workers with the realization that they cannot directly observe the workers' productivity S except the workers' education levels y. So the employers set the wage $W(y)$ according to the education level. The workers' productivity is determined by the innate ability n and acquired education level y, i.e., $S = S(y,n)$. There is, however, a varying cost of acquiring education, i.e., $C = C(y,n)$. By assumption, we have $S_y > 0, S_n > 0, C_y > 0, C_n < 0$. The goal of the paper is to find out some sufficient conditions for the existence of a separating equilibrium so that workers can use education level as a signal to reveal their innate ability.

The optimization problem for workers is

$$\max_y W(y) - C(y,n),$$

and the first-order conditions are $W_y = C_y$ for each ability level n. In the same time, we know the labor market condition $W(y) = S(n,y)$ holds at the equilibrium.

Proposition: If $C_{yn} < 0$ and there exists y^o such that $S_y < C_y, \forall y \geq y^o$, then there exists a one-parameter family of equilibrium-offered wage schedules, all of which are equilibria. This one-parameter family is determined by $W_y = C_y$ and $W(y) = S(n,y)$.

Proof: It suffices to make sure that the second-order condition of the worker's optimization holds so that there exists an unique equilibrium education level corresponding to each ability level and wage schedule. That is, we need to make sure $W_{yy} - C_{yy} < 0$.

It is natural to examine how the equilibrium conditions vary by innate ability. Denote as y^* the equilibrium education level for a given wage schedule $W(y)$. How does y^* vary for each ability level n? We can totally differentiate the first-order conditions w.r.t. n, and have the following,

$$W_{yy} \cdot \frac{dy^*}{dn} - C_{yy} \cdot \frac{dy^*}{dn} - C_{yn} = 0,$$

which implies,

$$W_{yy} - C_{yy} = C_{yn} \cdot \frac{1}{dy^*/dn}.$$

Because the wage is set up so that $W(y) = S(n, y)$, how does the wage schedule change w.r.t. ability n? We can again totally differentiate this labor market condition to get

$$W_y \cdot \frac{dy^*}{dn} = S_y \cdot \frac{dy^*}{dn} + S_n,$$

which implies,

$$\frac{dy^*}{dn} = \frac{S_n}{W_y - S_y}.$$

Combining these two differentiation results and using the first-order conditions $W_y = C_y$, we reach the following,

$$W_{yy} - C_{yy} = C_{yn} \cdot \frac{C_y - S_y}{S_n}.$$

Since we have $S_n > 0$ by assumption, the second-order conditions hold if $C_{yn} < 0$ and $C_y > S_y$.

What exactly do these conditions mean? The condition $C_{yn} < 0$ means that the marginal cost of obtaining additional education is lower for more talented people. The condition $C_y > S_y$ means that the marginal cost of obtaining additional education exceeds the marginal productivity gain, i.e., the education level is more than enough comparing to the socially optimal level at which we have $C_{y^{so}} = S_{y^{so}} = W_{y^{so}}$. The feature of over-investment of education in this model bears a name of "dissipative signaling model" in economics or "money-burning model" in finance. The highly-talented people in this economy are making sacrifice by over-investing to reveal that they are indeed highly-talented. Note that if the signal, the education level in this case, is too productive socially, i.e., $C_y < S_y$, then we won't reach an equilibrium in that the education level cannot effectively separate the talented from the less talented.

Now we know that the employers' objective is to design the wage schedules $W^*(y)$ such that $W^*(y) = S(y, n)$, $W^*_y = C_y$ and $W^*_{yy} - C_{yy} < 0$ all hold. The workers then would choose an optimal education level y^* that truly reflects their individual ability.

Corollary: In any equilibrium mentioned above, it must be the case that $W_y > S_y$, i.e., the contractual marginal rewards is greater than the socially optimal level S_y and that $dy^*/dn > 0$, i.e., more talented people choose higher education level.

Proof: We can easily use the first-order conditions $W_y = C_y$ and one of the conditions required, $C_y > S_y, \forall y \geq y^o$, to reach the first conclusion above immediately. Similarly, we have

$$\frac{dy^*}{dn} = \frac{S_n}{W_y - S_y} = \frac{S_n}{C_y - S_y} > 0. \text{ QED}$$

The finding of $W_y > S_y$ reveals the fact that employers pay higher wages to better-educated not only because the higher productivity associated with higher education, but also because a higher education signals higher capability.

It is useful to find out the full-information counterpart to compare the results. The optimization problem here is $\max_y W(y) - C(y, n)$. Since the workers' abilities n are always observable, the condition $W(y) = S(y, n)$ holds all the time, not just at the equilibrium. Therefore, we can replace the $W(y)$ in the objective with $S(y, n)$, i.e., to find a solution for $\max_y S(y, n) - C(y, n)$. The social optimal solution y^{so} satisfies $C_{y^{so}} = S_{y^{so}} = W_{y^{so}}$, as mentioned earlier. The comparison of y^{so} under full information and y^* under asymmetric information is depicted in the figure below.

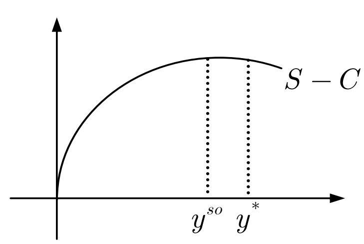

Figure 1.2: First- and Second-best Solutions in Spence Model

1.3 References

- Akerlof, G. A., 1970, "Market for Lemons - Quality Uncertainty and Market Mechanism," *Quarterly Journal of Economics*, 84, 488-500.
- Spence, M., 1974, "Competitive and Optimal Responses to Signals - Analysis of Efficiency and Distribution," *Journal of Economic Theory*, 7, 296-332.

Chapter 2

Self-selection Models

2.1 Rothschild and Stiglitz (1976)

The insurance market has potentially huge adverse selection, problems that could lead to market breakdown. But in reality we still observe the insurance market functioning. The research question here is: can we design an equilibrium model in which an equilibrium exists despite adverse selection? Moreover, can this equilibrium be unique?

Assume for now that there is only one type of individuals whose income is W when no accident happens. The individuals would enjoy income $W-d$ when an accident does happen with probability p. There are many insurance companies offering insurance contracts with different premium α_1 and corresponding policy claim $\hat{\alpha}_2$. The individuals are assumed to be risk-averse and insurance companies are risk-neutral. Denote as W_1 the individuals' income in the case of no accident, i.e., $W_1 \equiv W - \alpha_1$, and denote as W_2 the individuals' income in the case of accident, i.e., $W_2 \equiv W - d - \alpha_1 + \hat{\alpha}_2 \equiv W - d + \alpha_2$.

On the demand side for insurance, the individuals are maximizing the expected utility in the form of $V(p, W_1, W_2) = (1-p)U(W_1) + pU(W_2)$, where $U' > 0$ and $U'' < 0$. The value of the insurance contract (α_1, α_2) can be expressed as $V(p, \alpha_1, \alpha_2) = V(p, W - \alpha_1, W - d + \alpha_2)$ to the individuals. So the optimization problem for individual with risk probability p is to choose (α_1, α_2) to maximize the expected utility. Here we assume that the individual rationality condition[1] holds, i.e., $V(p, \alpha_1, \alpha_2) \geq V(p, 0, 0) \equiv$

[1] Note that the individual rationality (IR) condition is one of two constraints used very often in analyzing asymmetric information models. The other constraint also used very

$V(p, W, W-d)$, i.e., holding the insurance policy should make individuals no worse off than not holding the policy. On the supply side of insurance, the insurance companies are trying to maximize the expected profits, $\pi(p, \alpha_1, \alpha_2) = (1-p)\cdot\alpha_1 - p\cdot\alpha_2$. Since we are going to use the incomes in two cases, W_1 and W_2, as the sorting variables[2], we can express the profit function in space of sorting variables. That is,

$$\pi = \alpha_1 - p\cdot(\alpha_1+\alpha_2) = W - (1-p)\cdot W_1 - p\cdot W_2 - p\cdot d.$$

By assumption, all insurance companies are risk-neutral. The free entry to the markets will force the expected profit to insurance company to be zero, and thus

$$\pi = 0, \text{ or } (1-p)\cdot W_1 + p\cdot W_2 = W - d$$

will be the zero-profit line (ZPL) for the insurance companies. Furthermore, we notice that the slope of the ZPL is $dW_2/dW_1 = -(1-p)/p$. So we know that the ZPL is downward sloping and that the magnitude of the slope of the ZPL depends on the risk level of the individuals of getting into an accident.

Definition: The Rothschild-Stiglitz Nash Equilibrium (RSNE) is defined to be a set of insurance contracts such that when consumers choose contracts to maximize expected utility: (1) no contract in the equilibrium set makes negative profit for the insurance companies; (2) no contract outside the equilibrium set, if offered, will make a positive profit for the insurance companies.

This definition may seem very natural, but it turns out that it is very restrictive. As for now, we can solve for the first-best solution to the model with symmetric information. The fair price corresponding to the zero-profit condition is

$$\frac{p}{1-p} = \frac{\alpha_1}{\alpha_2} \text{ or } p = \frac{\alpha_1}{\hat{\alpha}_2},$$

i.e., the price of the insurance should be the same as the subjective probability of accidence occurrence. The individuals' optimization problem becomes

$$\max_{\alpha_1} pU(W - d - \alpha_1 + \alpha_1/p) + (1-p)U(W-\alpha_1).$$

often is the so-called "incentive compatibility (IC) constraints," which essentially says that agents are truth-telling.

[2] In any asymmetric information model, we need at least two sorting variables to work with. Note that the sorting variables must be something observable by all players and a good choice of sorting variables will make the interpretation of the model very intuitive.

The first order condition is

$$p \cdot (1/p - 1) \cdot U'(W - d + \alpha_2) - (1 - p) \cdot U'(W - \alpha_1) = 0, \text{ or}$$

$$U'(W - d + \alpha_2) = U'(W - \alpha_1) \text{ or } W_1 = W_2.$$

Alternatively, we can show that regardless of insurance holding α_1, the individuals' expected wealth is

$$p(W - d - \alpha_1 + \alpha_1/p) + (1 - p)(W - \alpha_1) = W - pd.$$

The full insurance $\alpha_1 = pd$ can achieve this expected wealth with certainty, and risk-aversion leads to full insurance, i.e., $W_1 = W_2$. That is, the equilibrium will be the intersection point of the 45^0 line (so that the expected income will be certain and equal across states[3]) and the ZPL with slope $-(1 - p)/p$. One such equilibrium is depicted in the following figure. Note that moving towards northeast would increase the individuals' iso-utility curve and reduce the insurance companies' profit level.

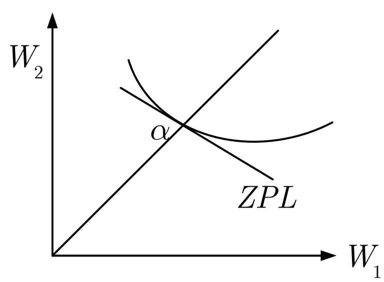

Figure 2.1: State-Wealth Space in Rothschild-Stiglitz Model

Now let's introduce asymmetry into the model. Assume that each individual can be either of two types, high-risk type with p^H or low-risk type with p^L, where $p^H > p^L$. The population proportion of high-risk types is λ. Assume further that each individual recognizes his/her own type but the insurance companies don't. The uninformed party, the insurance companies, is assumed to move first in this model by providing insurance polices for the informed party to choose from. Let's denote the average or pooling risk probability of involving an accident in the population as $\overline{p} \equiv \lambda \cdot p^H + (1 - \lambda) \cdot p^L$.

[3] To get this result, substitute the insurance companies' zero profit condition into the first-order condition for customers' optimization problem.

The expected utility function will be

$$V_i(p^i, W_1, W_2) = (1 - p^i) \cdot U(W_1) + p^i \cdot U(W_2),$$

where $i \in \{L, H\}$. We can use the implicit function theorem to find out the slope of the iso-utility curve as

$$\left(\frac{dW_2}{dW_1}\right)^i = -\frac{\partial V_i / \partial W_1}{\partial V_i / \partial W_2} = -\frac{(1 - p^i) \cdot U'(W_1)}{p^i \cdot U'(W_2)}.$$

We know that these iso-utility curves have negative slopes at all points and for both types. But we would like to know which one is steeper. So we find out

$$\left|\frac{dW_2}{dW_1}\right|^{i=H} - \left|\frac{dW_2}{dW_1}\right|^{i=L} = \frac{U'(W_1)}{U'(W_2)} \frac{p^L - p^H}{p^L p^H} < 0,$$

and conclude that the low-risk types have steeper iso-utility curves at each point on the space of sorting variables. In the same time, we know that the low-risk types also have steeper ZPLs.

It is intuitive to note that in this model it is the high-risk types are jeopardizing, and thus "bad guys," the low-risk types in that the latter could have gotten more favorable insurance contracts if there were no high-risk types around. However, it is true here and in every asymmetric information model that the "bad guys" end up getting the first-best solutions in the full information counterpart. Why? The high-risk types are no better off than they would be in the absence of low-risk types if there exists a successful separating equilibrium. So we know the high-risk types will be at the intersection point α^H of the 45^0 line and the high-risk ZPL in the equilibrium in the next figure.

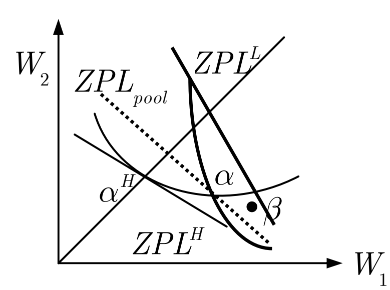

Figure 2.2: A Pooling Equilibrium Is Never a RSNE

Note further that a pooling equilibrium is not possible in this model. Suppose the contrary is true and the pooling equilibrium is indeed the intersection point α of the pooling ZPL and the high-risk types' iso-utility curve through the point α^H, in the figure above. Then we find that any insurance contracts lying between the iso-utility curves of two types to the right of point α, for example point β that lies below the low-risk ZPL, will not be interesting to high-risk types yet more attractive to low-risk types who are currently at point α. Moreover, since this contract attracts only the low-risk types (a phenomenon also known as "cream-skimming," as opposed to "adverse selection."), the insurance company provides the contract β clearly will make positive profit[4], violating the condition (2) in the RSNE. Hence the pooling equilibrium α is not a RSNE.

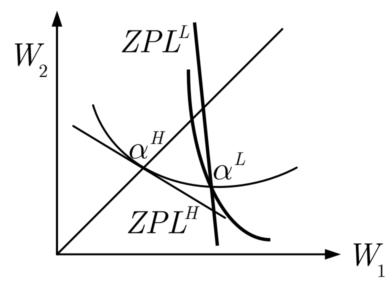

Figure 2.3: A Separating Equilibrium Is Possibly a RSNE

Now let's look at just the separating equilibrium described in the figure above. Again, we know the high-risk types will stay at α^H. It becomes natural[5] to see that the intersection point α^L of the high-risk types iso-utility curve through point α^H and the low-risk ZPL is where the low-risk types will rest. Is the combination (α^L, α^H) a RSNE indeed? We need to impose another constraint here, namely the incentive compatibility constraint. It

[4] Note that although the contract β lies above the pooling ZPL, the insurance company offers it would make positive profit. It may seems initially that it is against the intuition that moving towards southwest will increase the profits for insurance company, but the pooling ZPL is not relevant any more here since the contract β, relative to α, attracts only low-risk types. The pooling ZPL is meaningful only if two types are participating.

[5] The contract α^L to the low-risk types must satisfy the following three conditions to qualify for an equilibrium: (1) α^L must be on or below the high-risk type iso-utility curve through α^H so that high-risk types won't mimic; (2) α^L must be on the low-risk type ZPL to ensure that the insurance companies make zero profit by providing the contract α^L; (3) α^L must achieve maximum utility level feasible for the low-risk types.

is basically saying that the low-risk types will be willing to stay at α^L and the high-risk types will be willing to stay at α^H, and thus effectively reveal their true risk types to the insurance companies.

How do we know? We have to compare the separating result with the pooling result which doesn't qualify for a RSNE. That is, we need to consider the location of the pooling ZPL. It turns out that as long as the pooling ZPL doesn't intersect the low-risk types iso-utility curve through α^L, the combination (α^L, α^H) is a RSNE. Let's consider the case where the pooling ZPL does intersect the low-risk types iso-utility curve through α^L as in the figure below.

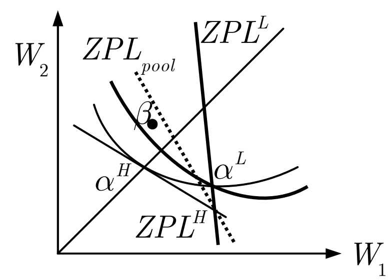

Figure 2.4: Existence of a Separating RSNE

In this case, clearly any contracts lying between the pooling ZPL and the low-risk types iso-utility curve through α^L and to the left of the intersection point of these two schedules, such as point β, will be attractive to both types since it is to the northeast of two iso-utility curves relevant here. In the meantime, the insurance company providing the contract β also makes positive profit because it attracts both types of agents and lies below the pooling ZPL. Under this circumstance, the combination (α^L, α^H) is not a RSNE since the contract β violates the condition (2) in the definition of RSNE.

It is now time to relate this paper to the Spence paper. How come the Spence paper offers infinite amount of equilibria[6] while this paper provides only one equilibrium, if it exists at all? It turns out that it's due to the

[6] In the figure where the pooling ZPL doesn't intersect the low-risk types iso-utility curve through α^L, the Spence corresponding separating equilibria are any contract combination with the same α^H and any points below α^L on the low-risk types ZPL through α^L.

difference in setup: although the labor market equilibrium condition $W(y) = S(y, n)$ in Spence is similar to the zero-expected-profit condition in R-S, there are infinite ways to pay wages to the workers in Spence, and workers are not restricted to pick the wage that maximize the workers' utility.

Note that in both the Spence model and the R-S model, the uninformed agent moves the first. In this type of game, the precommittment by the uninformed is critical. Although there is a strong incentive for the uninformed agent to re-negotiate the contract with the better-type among the informed agents, they cannot afford doing so. Otherwise, the worst-type of informed agents will no longer be willing to take his first-best solution and effectively spoil the equilibrium outcome.

Another interesting, maybe ironic as well, point is that as more and more low-risk types individuals exist in the market, the pooling ZPL will be closer and closer to the low-risk types ZPL and thus it becomes more likely that the pooling ZPL will intersect the low-risk types iso-utility curve through the point α^L. As a consequence, it is very likely there will be no RSNE at all. It may seems counter-intuitive at first, but the real reason behind is that as more and more low-risk types individuals exist in the market, it becomes harder and harder to adopt a separating equilibrium, i.e., the cost of separation becomes higher and the pooling equilibrium looks more attractive.

One important and very realistic lesson from the Rothschild-Stiglitz model is that de-regulation leads to product differentiation as a pooling equilibrium doesn't exist in a competitive market. Here is an example on the banking industry. Before the de-regulation, banks didn't have to pay interest on deposits yet offered cash-management services for free, i.e., banks used profits from deposits to subsidize the cash-management services. Once the de-regulation was implemented, this type of bundling practice didn't work anymore because many customers who didn't need cash-management services switched to banks that offered interest on deposits. Another relevant example is the implementation of AT&T's latest strategy to bundle cable services with consumer long distance services, in an apparent attempt to make up for the ever deteriorating margin on the long distance market. This strategy failed simply because other companies came along providing non-bundled services at more competitive prices.

How should we handle an asymmetric information model with more than two types? We sort those types by adversity. And the worst types will get the first-best solution in his full information counterpart, and his iso-utility curve through that solution will then intersect the ZPL for the next bad types at the solution for the next bad types, and we worked this way up until we reach the best types. One thing to note is though when there are more than two types in the model, it becomes more and more likely that there will be no RSNE at all due to the higher likelihood that the pooling ZPL will intersects the higher types iso-utility curves. A continuum of types would certainly guarantee the non-existence of RSNE.

2.2 Riley (1979)

One of the major results established in Riley's paper is that if the unknown parameter θ, that captures the underlying characteristics of the informed agents, lies in a continuum $[\underline{\theta}, \overline{\theta}]$, then there doesn't exist a RSNE. As we mentioned above, it is readily seen that this is the case because the pooling ZPL will intersect the ZPL of the best types for sure. The natural research question then becomes: can we refine the RSNE equilibrium concept in such a way that makes the existence of equilibrium possible?

Definition: A set of offers are Riley Reactive Equilibrium (RRE) if for any additional offer that generates an expected gain to the deviating agent i making the offer, there exists another offer that can be made by agent j that produces positive profits for j and negative profits for i. Moreover, there exists no further offer such that j can be made to suffer losses.

Here is the translation of the definition. Suppose we have only two types of agents in the model. If type i were to defect by making an offer different from the equilibrium solution, then type j could make another offer so that type i would be surely worse-off than the original equilibrium solution and type j would be better-off than the equilibrium solution. Moreover, any other offer couldn't make type j worse-off than the original equilibrium solution specified for type j. In retrospection, type i would realize that it's not worthwhile to defect in the first place, and thus the original equilibrium is preserved and called a Riley Reactive Equilibrium.

Theorem: There exists a unique RRE that is the Pareto dominating member of the set of strongly informationally consistent contracts (i.e., perfectly separating ones).

A crucial implication of the Riley's work is that RRE makes equilibrium existence possible when RSNE does not exist, but RRE does not change the equilibrium when RSNE does exist.

We are going to discuss only the six assumptions used in the Riley's paper.
(A1) $\theta \in [\underline{\theta}, \overline{\theta}]$ for strictly increasing, continuously differentiable $f(\theta)$, and $y \in [\underline{y}, \overline{y}]$.
(A2) The utility to the informed agent is $U(\theta; y, p)$, where θ is the informed agents' attribute that is unknown to the uninformed, y is the signal and p is the price of the underlying product such as wage in the labor market. The utility to the uninformed agent is $V(\theta; y)$. We assume that both U and V are continuously differentiable.
(A3) $U(\theta; y, p)$ is strictly increasing in p, i.e., the higher price, the better-off the seller will be.
(A4) $V(\theta; y) > 0, V_1(\theta; y) > 0$ and $V_2(\theta; y) \geq 0$.
(A5) $\frac{\partial}{\partial \theta}\left(-\frac{U_2}{U_3}\right) < 0$, where $U_2 = \partial U / \partial y$ and $U_3 = \partial U / \partial p$.
(A6) $\forall \theta \in \Theta$, $U[\theta; y, V(\theta; y)]$ is either strictly decreasing in y or it has a unique turning point at $y^*(\theta)$ which is the utility maximum level of the signal. Moreover, $\forall \theta \in \Theta$, we have $U[\theta; \overline{y}, V(\theta; \overline{y})] < U[\theta; y^*(\underline{\theta}), V(\underline{\theta}; y^*(\theta))]$, where $y \in [\underline{y}, \overline{y}]$ and $\theta \in [\underline{\theta}, \overline{\theta}]$ as in (A1).

The first part of the assumption (A4) says that all products have positive values to the buyer. The second part says that a higher value of θ represents a better product. The third part says that a higher value of signal y will not be viewed negatively by the buyer. The assumption (A5) is the most crucial one in this model and we often call it the "*single crossing property.*" If we totally differentiate the informed agent's utility function at a fixed level w.r.t. y, we have

$$U_2(\theta; y, p) + U_3(\theta; y, p)(dp/dy) = 0, \text{ or } \frac{dp}{dy} = -\frac{U_2}{U_3} = -\frac{\partial U / \partial y}{\partial U / \partial p}.$$

Since $U_2 < 0$ and $U_3 > 0$, we have $dp/dy > 0$. What the assumption (A5) says is that $\partial(^{dp}/_{dy})/\partial\theta < 0$, i.e., high-quality sellers have flatter iso-utility curves in the (p, y) sorting space.

The assumption (A6) is saying that, if U is strictly decreasing in y, the first-best solution is $y^* = 0$. This assumption effectively avoids corner solution for the separating equilibrium as the upper corner $\overline{y}$ yields utility level even less than the worst type's. Note that if we have multiple dimensional signals (i.e., more than two types of signals), we don't need this assumption.

In the Rothschild-Stiglitz context, we have $\theta \in \{\theta^H, \theta^L\}$ and a higher value of θ represents bad attribute. So the lower-risk types have steeper slopes in the (W_1, W_2) space, consistent[7] with the assumption (A5).

We can prove that the assumption (A5) implies that $C_{yn} < 0$ in the Spence's context, but the converse is not true unless the informed is risk-neutral. First of all, note that the ability level n and wage W in Spence model are equivalently θ and p in Riley's notation.

(a) Assumption (A5) in Riley implies $C_{yn} < 0$ in Spence.

For the utility function $U(n; y, W) = W(y) - C(y, n)$, if we totally differentiate the iso-utility curve at a fixed level with respect to y, we get the following,

$$W_y - C_y = 0, \text{ or } C_y = W_y.$$

Since in Riley's notation we have $W_y = dp/dy$, we easily get

$$\frac{\partial}{\partial \theta}\left(\frac{dp}{dy}\right) = \frac{\partial}{\partial n}(C_y) = C_{yn} < 0, \text{ by (A5).}$$

(b) $C_{yn} < 0$ in Spence can easily imply (A5) in Riley if the workers are risk neutral. If workers are risk averse, however, we cannot get the result. Here is why. Assume that the worker's utility function is separable and concave, in the form of

$$U(n; y, W) = U[W(y)] - V[C(y, n)].$$

If we totally differentiating the iso-utility curve with respect to y, we get

$$U'W_y - V'C_y = 0, \text{ or } W_y = \frac{V'}{U'}C_y,$$

so that

$$\frac{\partial}{\partial \theta}\left(\frac{dp}{dy}\right) = \frac{\partial}{\partial n}(W_y) \neq C_{yn},$$

i.e., the condition $C_{yn} < 0$ doesn't imply (A5) in Riley.

[7] You don't think they are really consistent? Recall that we have downward sloping iso-utility curves in the (W_1, W_2) space in RS model but upward sloping iso-utility curves in the (p, y) space here.

2.3 Ofer and Thakor (1987)

Empirical evidence suggests that when a public firm announces a dividend payment, there is generally an announcement effect measured by cumulative abnormal return of between 0.85% and 1.39% on average, yet the cumulative abnormal return associated with a tender offer of stock repurchase is about 17.75% on average. What's the difference between a dividend and stock repurchase as a signal of the firm value, and why there is such a huge difference between the impact of these two types of signals? Ofer and Thakor try to shed some light on this important question from a theoretical perspective.

At time $t = 0$, a manager who owns α of a firm is given a wage contract W that is the market value of b shares of stock at $t = 0$. The wage is paid out in period $t = 1$. The manager announces at $t = 0$ the policy for paying out cash at time $t = 1$. The manager may choose to pay dividend d, repurchase (tender-offer repurchase) stocks worth β of the firm, or both. The manager is assumed to exclude himself/herself from the stock repurchase plan, something supported by empirical evidences. The firm has a project at time $t = 0$ that will produce interim cash flow at time $t = 1$ and terminal cash flow at time $t = 2$. The interim cash flow is C with probability $1 - \xi$ and 0 with probability ξ. The terminal cash flow $\tilde{\pi}$ is random with mean π and variance σ^2. If the interim cash flow C is realized at time $t = 1$, it can be used for either dividend d or repurchase β or both, after paying the wage W. In the case where the interim cash flow falls short, the firm is obligated to borrow from outside at a punitive rate R which is greater than the risk-free rate r. No tax is considered in this model. Investors are assumed to be risk-neutral and the manager risk-averse.

Before we build up the model, let's think about what the potential costs involved in the policy announcement may be. Clearly, the firm is subject to the distress financing cost should the interim cash flow fall short of the target, and the manager is subject to a loss of wage from a lower stock price. In the same time, it is very important to note that the manager also suffers from another source of cost if the repurchase plan is adopted. That is, the manager will be exposed to more risk since the self-exclusion from the tender offer makes the manger have a larger share, $\alpha/(1-\beta)$, of a more risky firm, for some of the existing cash flow is used to buy back shares and thus the weight on the risky terminal cash flow is even larger.

One obvious difference between the model in this paper and the Riley model is that here we are discussing two signals, dividend and repurchase, whereas the Riley model has only one signal. The immediate benefit of having two signals is that we can now evaluate the simultaneous signals in Pareto optimal order. In both of these two papers, there is only one unknown attribute discussed. If we were to extend the number of attributes[8], it will become very hard to rank the cross-attribute combinations, let alone justifying the rank.

Now let's introduce the information asymmetry. The firm could be either a good type or a bad type. For simplicity[9], we assume that $\xi_g = 0, \xi_b = 1$, $\pi_g > \pi_b > 0$ and $\sigma_g^2 = \sigma_b^2$. So here the good type firm will have better cash flows at both periods. The manager knows the firm type and thus its interim cash flow but the investors don't. Moreover, $\pi_g, \pi_b, \sigma_g^2, \sigma_b^2$ are public information. The manager is assumed to have mean-variance utility, $E_i - k \cdot \sigma_i^2$ for $i \in \{g, b\}$, where E_i is the manager's expected payoff from wage and market value of stocks held at time $t = 0$.

Proposition 1: The first-best solution for the manager's announcement is $d^* = 0$ and $\beta^* = 0$, provided that $0 < \xi_g < \xi_b < 1$. The second-best solution for the manager's announcement is a Riley Reactive Equilibrium (RRE), i.e., a set of signals and prices $\Lambda \equiv \{(d_g^*, \beta_g^*, V_g^*), (d_b^*, \beta_b^*, V_b^*)\}$ such that for any additional set $\mathcal{S}$ that produces profits for investors (e.g. buy the stocks for less than its worth), when $\mathcal{S} \cup \Lambda$ is offered, there exists an additional set $\mathcal{S}'$ such that $\mathcal{S}$ generates losses for investors and each contract in $\mathcal{S}'$ is strictly profitable for the manager when $\mathcal{S} \cup \Lambda \cup \mathcal{S}'$ is offered. Moreover, the set $\mathcal{S}'$ also makes the manager immune to losses.

Why would the manager's first-best solution be to announce no dividend and no stock repurchase? Because there is a positive probability that the firm has to borrow at a punitive rate R under either dividend or repurchase policy, and particularly the repurchase policy would add additional cost to the manager due to a higher exposure to a riskier company after the repurchase. Note that the conditions in a RRE would make the possible defector, the uninformed investors in this case, feel that it's not worthwhile to defect in the first place and thus the equilibrium is preserved.

One important feature shared by all models we have studied so far is that the uninformed agent make a pre-commitment to abide by the contract

[8] For a nice attempt to extend the number of unknown attributes, read Baron and Heyesson (1982), "Regulating a Monopolist with Unknown Costs," *Econometrica*.

[9] Had we specified a different set of values for ξ_g and ξ_b while keeping the order $\xi_g > \xi_b$, the results won't change except that we would have to carry out the expected interim cash flow for two types throughout the model, not like C and 0 under our assumption here. We insist imposing $\xi_g > \xi_b$ so that the good type firm is unambiguously better than the bad type.

agreed. We will notice later on that in models with informed agent moving first, the pre-commitment by the uninformed agent is not necessary.

Proposition 2: If no repurchase is allowed, then there exists $d > 0$ such that $d_g^* > 0,\ d_b^* = 0$ as long as $d_g^* \le C - W_g$.

The condition $d_g^* \le C - W_g$ makes sure that the good type firm will not incur distress financing cost, and thus the total signaling cost is zero. If the bad-type firm decides to mimic the signal, however, it will surely incur positive signaling cost. So $d_g^* > 0,\ d_b^* = 0$ would be a separating equilibrium.

Note that for given level of $\pi_g - \pi_b$, the price effect of dividend policy is zero for the good type firm because the good type firm has enough interim cash flow to finance the dividend payment d_g^*, and the swap between cash flow and dividend won't change the firm's valuation. By mimicking the good type firm's dividend policy, the bad type firm's stock will have a temporary jump from the cross-sectional average price at $t = 0$ but fall sharply at $t = 1$. In the meantime, the signaling cost the bad type firm incurs will increase as d_g^* becomes higher. Therefore, as long as d_g^* is high enough, the bad type firm won't afford mimicking the good type firm's dividend policy.

Proposition 3: As long as $d_g^* \le C - W_g$, we have $\beta_g^* = \beta_b^* = 0$.

Why both types of firms won't choose the repurchasing plan? Because the repurchase plan expose managers to additional risk due to the larger share of the riskier assets they hold. As a matter of fact, the magnitude of $\pi_g - \pi_b$ determines whether $d_g^* \le C - W_g$. Using the incentive compatibility constraint, we find that $\partial d_g^* / \partial(\pi_g - \pi_b) > 0$.

Proposition 4: If $\pi_g \gg \pi_b$, then a repurchase plan is optimal, possibly in conjunction with a dividend policy.

That is to say, if π_g is sufficiently better than π_b, so that the bad type firm won't afford mimicking, and more importantly if the amount the good type firm spent on repurchase is less than $C - W_g$, then the good type firm manager will do both the repurchase plan and dividend policy, i.e., $\beta_g^* > 0$ and $d_g^* > 0$.

Since we have two simultaneous signals available in this case, we can compare the relative efficiency of the two signals, from both the perspective of aggregate signaling cost and that of marginal signaling cost.

Let's discuss the effectiveness of the dividend policy at first. (1) When $d_g^* < C - W_g$, the dividend policy will do a good job of separating types in terms of not only aggregate signaling cost, but marginal cost as well. The former is true because the aggregate signaling cost is zero for the good type and positive for the bad type. The latter is true because the marginal signaling cost is also inversely related to firm's quality. (2) When $d_g^* > C - W_g$, the dividend won't do an as efficient job as in the previous case. Both types of firms will be subject to distress financing cost if they adopt the policy d_g^*, although the aggregate signaling cost is higher for the bad type firm than the good type firm. Moreover, the marginal signaling cost is the same across two types.

How effective is a repurchase plan to separate types then? The signaling cost is *always* higher in both the aggregate terms and the marginal terms for the bad type firm than those for the good type firm, if a stock repurchase plan is adopted. The reason behind is that despite the same risk across two types of firms, i.e., $\sigma_g^2 = \sigma_b^2$, when a repurchase plan is adopted, the good type firm is worth more than the bad type firm on average, i.e., $\pi_g > \pi_b$. Therefore, the additional cost to the manager from a repurchase plan will be higher in aggregate terms in the case of a bad type firm than in the case of a good type firm. Since a higher β_g^* will increase the effective share of the manager in the firm and thus a higher risk exposure, the signaling cost of a repurchase plan will be also inversely related to the firm quality in the marginal sense.

Once we have developed the idea that a stock repurchase plan is a more powerful, or efficient, signal than the dividend policy, it is easy to realize that if the amount the good type firm spent on the stock repurchase plan is less than $C - W_g$, then the good type firm can use the difference to impose the additional signal of a dividend policy so that the mimicking cost will be so huge as to deter the bad type firms from deviating from its first-best solution, i.e., sending zero-valued signals.

Proposition 5: If the amount the good type firm spent on repurchase is greater than $C - W_g$ while $\pi_g \gg \pi_b$, then there will be no dividends paid out at all.

The reasoning behind the proposition above is fairly simple. Even the more efficient signal, the stock repurchase plan, will force the good type firms to borrow from outside at a punitive rate under this special case. The use of a dividend policy could only do an inferior job to separate the two types and thus is not something sensible to do.

As the figure above depicts, when the mean difference of terminal cash flows across two types locates in region I, the good type firm sends only the signal of dividend policy; in region II, the good type firm sends both signals; and in region III, the good type firm uses only the signal of stock

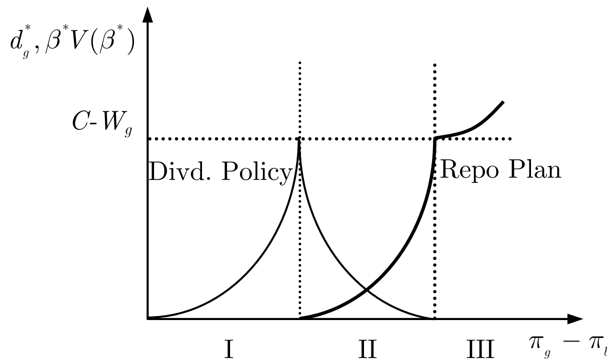

Figure 2.5: When to Use Dividend/Stock Repurchase?

repurchase.

Now let's depict the equilibrium in the following graph for the case where $\pi_g - \pi_b$ is not very large. We use the market value of the firm at time $t = 0$, $V_i(d), i \in \{g, b\}$, and the dividend policy announced as the two sorting variables in this model.

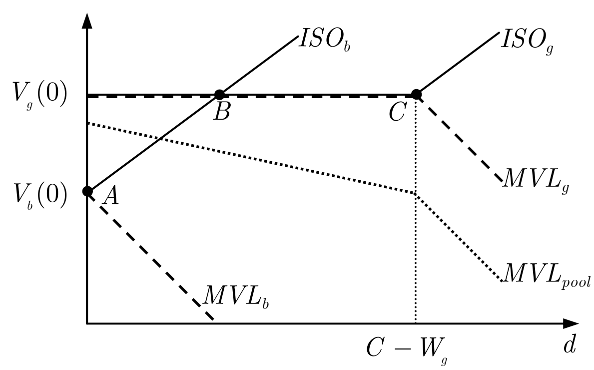

Figure 2.6: A Separating Equilibrium that is both RSNE and RRE

It is clear that the northwest direction is the direction of increasing utility level for both types of firms and that any point below the appropriate market value line will bring positive profits to investors. Since the signaling cost for the good type firm is positive for $d > C - W_g$, both the utility for the manager and the market value of the firm should fall, hence the iso-utility curve diverges from the market value line.

Here are a couple of quick facts. First of all, as we have already found many times by now, the bad type firm occupies point A which is its first-best solution. Second, the bad type iso-utility curve (ISO_b) through point A intersects the good type market value line (MVL_g) at point B. Third, the highest feasible iso-utility curve for the good type (ISO_g) that intersects the MVL_g is determined correspondingly. It is obvious that any point along the ISO_g and between point B and point C will be an equilibrium point for the good type firm. We can safely say that the combination of (A, B) is a RSNE and thus RRE.

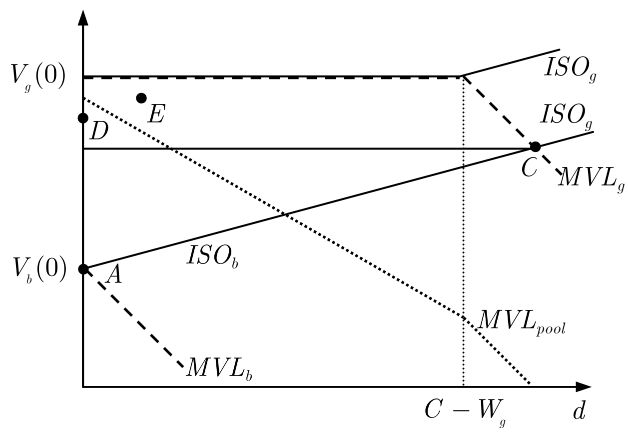

Figure 2.7: A Separating Equilibrium that is not RSNE but RRE

Let's consider the case where $\pi_g - \pi_b$ is sufficiently large. Let's go through the process of determining an equilibrium again. Without any question, the bad type will occupy its first-best solution, point A. Then the ISO_b through point A intersects the MVL_g at point C. The highest feasible ISO_g is determined correspondingly; it goes through point C and shares a portion with ISO_b. The issue at hand now is that we cannot say the combination (A, C) is a RSNE any more because the pooling MPL intersects the ISO_g through point C. To see why the combination (A, C) is not a RSNE, we take a note of the possible pooling equilibrium[10] point D that the bad type firm

[10] Why we should take care of the pooling equilibrium here, as we did before? Because we know the combination (A, C) is the best separating equilibrium, and the only type

manager could propose and defect from the combination (A, C). Clearly the point D gives both types of firm managers a higher utility level, and also makes investors earn positive profit because it locates under the pooling market value line. The point D violates the condition (2) in the definition of RSNE and thus disqualifies the combination (A, C) as a RSNE.

However, we would argue that the combination (A, C) is a Riley Reactive Equilibrium in the following sense. Once the bad type firm manager defects and proposes the point D, the good type firm manager could counter-offer with the point E. The point E makes the good type firm manager better off than the point D and attracts only the investors for the good type firm in that the point E lies way above the MVL_b and below the MVL_g. The fact that only investors for the bad type firm will be left at the point D drives the investors for the bad type firm leave the point D as well, because the pooling MVL is not relevant for point D any more and the point D lies above the MVL_b. Given that we have established the point E will make both manager and investors for the good type firm better-off and make investors for the bad type firm worse-off, and the fact that there is no other point that will make both manager and investors for the good type firm worse-off[11] than at the original separating equilibrium (A, C), the defector can only burn his/her own hands without being able to fight back. Therefore, the bad type firm manager won't defect from (A, C) and propose the point D in the first place. Therefore, (A, C) is preserved as a RRE.

One lesson from this exercise is that the privilege of defection is always reserved for the first-mover, the uninformed investors in this case. If the informed agents are allowed to move first, then the defector will be the uninformed but again the privilege of defection is on the side of the informed agents.

2.4 References

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of equilibrium that can break this best separating equilibrium is some kind of pooling equilibrium. (We don't consider mixed strategy equilibrium here.)

[11] The worst scenario for the manager and investors for the good type firm at point E will be that the positive profits investors are earning are wiped off, but they won't be worse-off than they are at the original separating equilibrium (A, C).

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Chapter 3

Game Theory

3.1 Some Definitions around a Game

3.1.1 Information Condition

Perfect Information: Each player knows where he's in the game, i.e., his information set at each stage contains of only one element. It is also known as "types and moves known." *Complete Information:* Player 2 knows that player 1 plays before him, but doesn't know player 1's actions. It is also known as "types are known and moves are unknown." *Incomplete Information:* Player 2 doesn't know which of players 1a or 1b moves, and doesn't know the action taken before him. It is also known as "types and moves are unknown."

3.1.2 Extensive Form vs. Strategic Form

An *extensive form* of a game requires the following: (1) physical order of play; (2) choice available to each player when it is his term to move; (3) rules to determine who moves when; (4) information a player has when he has to move; (5) payoffs to players as a function of moves; (6) initial conditions when the game starts. A *strategic form*, or *normal form*, of a game applies all the game information in a payoff matrix.

It is known that every game in extensive form can be reduced to one in a strategic form. Although many extensive forms reduce to the same strategic form, a given extensive form reduces to only one strategic form.

Note that some information will be lost upon reducing an extensive form into a strategic form.

Here are some notations on an extensive form. There is a finite set T of nodes t with a binary relation ¡ on T that represents precedence. The set of initial nodes is denoted as W, i.e., $W = \{t \in T | t \text{ has no predecessors}\}$. The set of terminal nodes is denoted as Z, i.e., $Z = \{t \in T | t \text{ has no successor}\}$. The set of decision nodes is denoted as X, i.e., $X = T \backslash Z$. The finite set of activities available to players is denoted as A. Each of the N players has a type τ. The history information set up to the current time is denoted as H. The set of feasible actions at node $x \in X$ is denoted as $\alpha(x)$. The collection $\{T, <, A, \alpha, N, T, H\}$ is an *extensive form.* If we add the following two things to the game: (1) player's utility to terminal nodes; (2) probabilities to initial nodes, then we get an *extensive game.*

3.1.3 Strategy

Strategy π_i for player i assigns to each information set $h \in H^i$ a probability measure on the action set $A(h)$, i.e., $\pi_i : A^i \to [0, 1]$. If these probabilities are degenerative, i.e., some actions have a probability of one, they are called *pure strategies*; otherwise, they are called *mixed strategies.* Although there is a slight difference between a "mixed strategy" and a "randomized strategy," we are going to use them interchangeably in this course.

3.1.4 Subform vs. Subgame

A *subform* of an extensive game is a collection of nodes $\hat{T} \subseteq T$, together with precedence relation ¡, types $\tau \in \overline{T}$, finite set of actions A, finite set of feasible actions $\alpha(x)$ at node x, and the history information set H, all defined on $\hat{T}$.

A *proper subform* is a subform $\hat{T}$ consisting *solely* of some node x and its successors. In this case, the initial node is a singleton, i.e., the probability that the subform starts at node x is one. In contrast, a *proper subgame* requires two things: (1) it is a proper subform (with a singleton initial node); (2) the subgame "inherits" all of the structure from the original game.

3.1.5 A Signaling Game

There are two players, A and B, in the signaling game. Player A, as the informed agent, moves first and the uninformed agent, player B, moves second. Player A's type $\tau \in \overline{T}$ is drawn according to some probability distribution ρ over $\overline{T}$ that is common knowledge. Player B's type doesn't matter here as long as every player's payoff function is common knowledge.

Player A sends player B a message $m \in M$, where M is a finite set and we have $M(\tau)$ in general. Player B then chooses a response $r \in R$, where R is also a finite set. Denote as $\overline{T}(m)$ the set of types that have available message m. The utility of player A is $u(m;\tau,r)$ and that of player B is $v(r;\tau,m)$.

The strategies for player A is $\phi(m;\tau) = \Pr(\tau$ will send message $m)$, and the strategies for player B is $\pi(r|m) = \Pr(B$ will repond to m with $r)$. Before the player A's move, player B has a belief about the player A's type; it is the common knowledge $\rho(\tau)$. After observing the message sent from player A, player B has a posterior belief about the player A's type, the probability distribution $\mu(\tau|m)$ over $\overline{T}(m)$, where $\overline{T}(m)$ is the set of types of player A that could have sent m.

Denote as $BR(\mu, m)$ the set of best responses by player B to message m, given posterior beliefs μ. Then it is obvious that

$$BR(\mu; m) = \underset{r \in R(m)}{\arg\max} \sum\nolimits_{\tau \in \overline{T}(m)} v(r;\tau,m) \cdot \mu(\tau|m).$$

If we allow the existence of mixed strategies in response to multiple $r \in BR(\mu; m)$ that produce the same expected payoff for player B, then we could denote the mixed best responses as MBR. From the definition above, it is easy to see that best responses are equivalent to posterior beliefs.

3.2 Nash Equilibrium

3.2.1 Definition of Nash Equilibrium

Note at first that a Nash Equilibrium amounts to specifying the strategies for all players involved that will sustain the equilibrium outcome.

In our signaling game above, player A's strategy ϕ is: given player B's strategy π, each type τ of player A evaluates the utility from sending message

m and receiving response r as

$$\sum\nolimits_{r \in R(m)} u(m;\tau,r) \cdot \pi(r|m),$$

and $\phi(m|\tau)$ puts weight on the message m if and only if it is among the maximizing m's in this expected utility. Similarly, player B's strategy π is: given player A's strategy ϕ, for any message m with positive probability by player A of some type τ, player B uses Bayes' Rule to compute the posterior assessment that m comes from $\tau \in \overline{T}(m)$ as

$$\mu(\tau|m) = \frac{\Pr(\tau,m)}{\Pr(m)} = \frac{\rho(\tau) \cdot \phi(m|\tau)}{\sum_{\tau' \in \overline{T}(m)} \rho(\tau') \cdot \phi(m|\tau')}.$$

The *Nash Equilibrium* (N.E. henceforth) condition is that for all message m sent by player A of some type τ with positive probability, every response r must satisfy

$$\pi(r|m) \in MBR(\mu(\tau|m);m).$$

A few points to note here. (1) A N.E. requires Bayesian rationality in equilibrium; (2) A N.E. imposes no restrictions on what player B should do when observing a message m with zero probability[1]. That is, the N.E. condition stipulates the strategies on the equilibrium path yet poses no restrictions on the strategies off the equilibrium path. As we will see soon in an effort to refine the N.E. concept, player B can pose threat regarding off-equilibrium strategies so as to sustain the N.E. outcome.

3.3 Fundamental Existence Theorem

For any strategic form game in which the set of players and the set of possible actions are all finite, there exists at least one Nash Equilibrium in randomized strategies. Moreover, whenever at least one player has a dominant strategy, there exists at least one pure-strategy Nash Equilibrium; if every player has a dominant strategy, there exists a unique pure-strategy Nash Equilibrium.

[1] If player B observes a message with zero probability, i.e., $\Pr(m) = 0$, or $\Sigma_{\tau' \in \overline{T}(m)} \rho(\tau') \cdot \phi(m|\tau') = 0$
then he cannot get the posterior belief appropriately using the Bayes' rule. The game designer has to specify the posterior beliefs for player B.

For example, the strategic form below produces no pure-strategy N.E.. It is assumed henceforth that the row player is player 1 and the column player is player 2. The mixed-strategy N.E. is $\pi_1(T) = \frac{3}{4}, \pi_1(B) = \frac{1}{4}, \pi_2(L) = \frac{1}{2}, \quad \pi_2(R) = \frac{1}{2}$. Note that the expected payoffs for two players under the mixed-strategy N.E. are $u_1 = 0$ and $u_2 = 0$. Although the pure-strategy (T, L) is not a N.E., it has the same payoff as the one under the mixed-strategy N.E.

	L	R
T	$(0,0)$	$(0,-1)$
B	$(1,0)$	$(-1,3)$

As a matter of fact, if we replace the payoff cell at (T, L) with (a_1, a_2), (B, L) with (b_1, b_2), (T, R) with (c_1, c_2), and (B, R) with (d_1, d_2), then the general solution for a mixed-strategy N.E. is

$$\begin{aligned} \pi_1(T) &= \tfrac{d_2-b_2}{a_2+d_2-b_2-c_2}, \pi_1(B) = \tfrac{a_2-c_2}{a_2+d_2-b_2-c_2}, \\ \pi_2(L) &= \tfrac{d_1-c_1}{a_1+d_1-b_1-c_1}, \pi_2(R) = \tfrac{a_1-b_1}{a_1+d_1-b_1-c_1}. \end{aligned}$$

And the expected payoffs are $u_1 = \frac{a_1 d_1 - b_1 c_1}{a_1+d_1-b_1-c_1}, u_2 = \frac{a_2 d_2 - b_2 c_2}{a_2+d_2-b_2-c_2}$.

3.3.1 Problems with Nash Equilibrium

Two of the most common problems with N.E. are (1) non-uniqueness and (2) inefficiency. Let's take a look at these problems in the following examples.

	L	R
T	$(5,5)$	$(0,6)$
B	$(6,0)$	$(\mathbf{1},\mathbf{1})$

In the game above, we have one unique N.E. (henceforth indicated by payoffs in bold face) that is not Pareto optimal, compared to the (T, L) combination.

	L	R
T	$(\mathbf{2},\mathbf{1})$	$(0,0)$
B	$(0,0)$	$(\mathbf{1},\mathbf{2})$

In the game above, we have two N.E.'s (in bold face), instead of one unique N.E.

3.3.2 Roadmap for Refining Nash Equilibrium

There have been three routes of refining N.E., as indicated in the following map. Before we focus our attention in the next section on the second route, i.e., "Sequential Rationality," we would briefly explain the first route below.

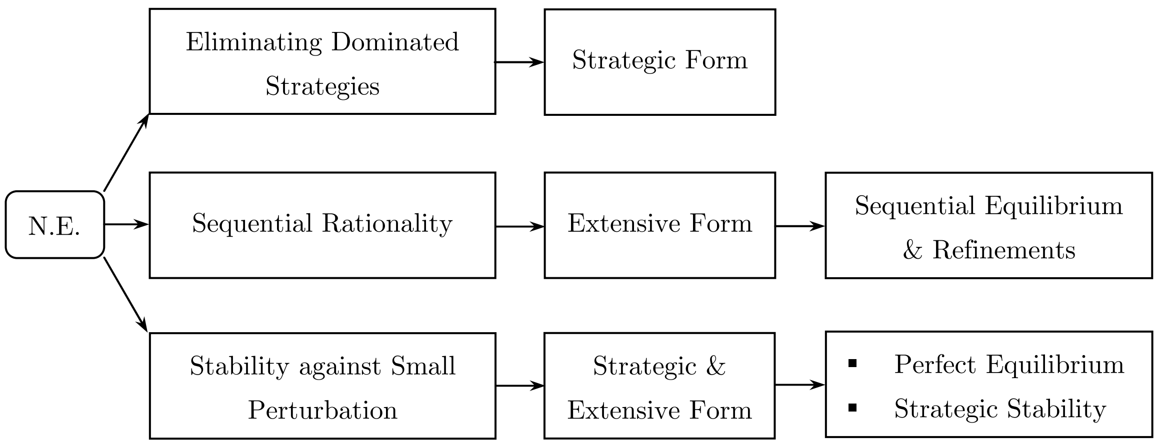

Figure 3.1: Roadmap for Refining Nash Equilibrium

Using the conventional two-player strategic form as

	L	R
T	$(\mathbf{1},\mathbf{9})$	$(1,9)$
B	$(0,0)$	$(\mathbf{2},\mathbf{1})$

we end up with two N.E.. Notice that the strategy L is weakly dominated by the strategy R. If we adopt the first route, eliminating the (weakly) dominated strategies, of refining N.E., we reach only one N.E., (B,R). This is certainly helpful. However, the elimination of (weakly) dominated strategies doesn't always lead to Pareto efficient equilibrium. For example, the following two-player strategic form

	L	R
T	$(\mathbf{5},\mathbf{5})$	$(0,5)$
B	$(5,0)$	$(\mathbf{1},\mathbf{1})$

has two N.E. once again. If we use the elimination of (weakly) dominated strategies, we end up with the (B,R) N.E., which is Pareto inferior to the (T,L) N.E.

Moreover, the sequence in which we eliminate dominated strategies may affect the final outcome when weakly dominated strategies are eliminated, but does not matter when strictly dominated strategies are eliminated. For example, the following strategic form

	x_2	y_2	z_2
x_1	$(3,3)$	$(0,3)$	$(0,0)$
y_1	$(3,0)$	$(2,2)$	$(0,2)$
z_1	$(0,0)$	$(2,0)$	$(1,1)$

would lead to different outcomes if we use different sequences of eliminating dominated strategies.

3.4 Subgame Perfection

A subgame qualifies to be a *proper subgame* if the following two conditions are met: (1) the initial node at which it is the player's turn to move is a singleton; (2) subgame inherits all of the structure of the original game. According to Selten, every strategy π is *subgame perfect* if for every proper subgame, the strategy π, restricted to the subgame, constitutes a Nash Equilibrium.

Let's consider the following game of extensive form.

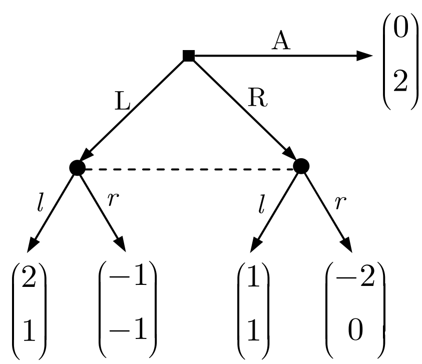

Figure 3.2: One Example of Sub-game Perfection

Here we have two pure-strategy N.E., $\{A, r\}$ and $\{L, l\}$. But the first N.E. is less plausible in that for player 2 the strategy l weakly dominates the strategy r. So the elimination of weakly dominated strategies could help us to reach one unique N.E. $\{L, l\}$. Yet we note that in this game, the only proper subgame is the whole game itself. That is, the N.E. $\{A, r\}$ is also subgame perfect. Why the notion of "subgame perfection" didn't help us here? It is precisely because of the nature of the information set in the game; the only proper subgame is the whole game itself. Note that generally "subgame perfection" won't help much in presence of information uncertainty.

3.5 Sequential Equilibrium

Note that "Sequential Equilibrium" is a subset of "Subgame Perfection Nash Equilibrium" which is a subset of "Nash Equilibrium." To require a Nash Equilibrium to be a "Sequential Equilibrium" we need to add the requirement that for every message m sent with zero probability, i.e.,

$$\forall m \text{ such that } \sum\nolimits_{\tau \in \overline{T}(m)} \rho(\tau) \cdot \phi(m|\tau) = 0,$$

there must be a probability distribution over types $\tau(m)$, written as $\mu(\tau|m)$, such that

$$\pi(r|m) \in MBR(\mu(\tau|m); m),$$

or if $u(m; \tau, r)$ is the utility of type τ from defecting with m when r is the (pure) best response of player B, given some belief μ about player A's type, then

$$u(m; \tau, r) \leq u^*(\tau),$$

for all $\tau \in \overline{T}(m)$ and the particular belief structure chosen in light of the out of equilibrium move.

The translation is that once player B observes a message m that is out of the equilibrium path, player B forms a posterior belief $\mu(\tau|m)$ about player A's type τ. Player B then sends back a best response r based upon the out of equilibrium path belief $\mu(\tau|m)$. The fact that the player A's out of equilibrium message m and the player B's response r to out of equilibrium moves makes the defector, player A, not better-off will sustain the original equilibrium as a Sequential Equilibrium (S.E.).

Application Rule: If we want to establish a sequential equilibrium, we need to specify only one out-of-equilibrium-path belief for player B that will deter player A from defecting, i.e., $u(m; \tau, r) \leq u^*(\tau)$ hold. On the other hand, if we want to prove that one N.E. is not S.E., we need to make sure that none of out-of-equilibrium-path beliefs for player B will make $u(m; \tau, r) \leq u^*(\tau)$ hold.

Properties of S.E.: (1) For every extensive form game with finite players and strategies, there exists at least one S.E.; (2) If (μ, π) is a S.E., then π is subgame perfect. When we are dealing with signaling games with single crossing property, the concept of "Bayesian Perfect Nash Equilibrium" is the same as the concept of "Sequential Equilibrium." But note that under general terms, these are two different concepts.

3.6 Applications to Sequential Equilibrium

3.6.1 One Simple Game

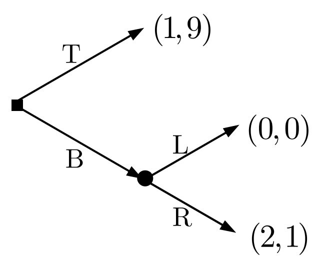

Figure 3.3: One Example of Sequential Equilibrium

As we have discussed before, there are two N.E. in this game, namely (T, L) and (B, R). Note that the notion of "subgame perfection" can help us to remove the N.E. (T, L) already. Similarly, we can also use the concept of "sequential equilibrium" to remove the N.E. (T, L). How? Suppose that the N.E. (T, L) is a S.E., then a defection by player 1 into playing B will force the player 2 to play his optimal response, that is R. According to the concept of S.E., the player 1's defection into $B$and player 2's best response R should make player 1 no-better-off relative to the equilibrium outcome (T, L), a fact obviously violated by the strategy (T, L). Hence, the N.E. (T, L) is not a S.E. On the other hand, the N.E. (B, R) is a S.E. in that a defection by player 1 into playing T will cause player 1 worse-off for sure.

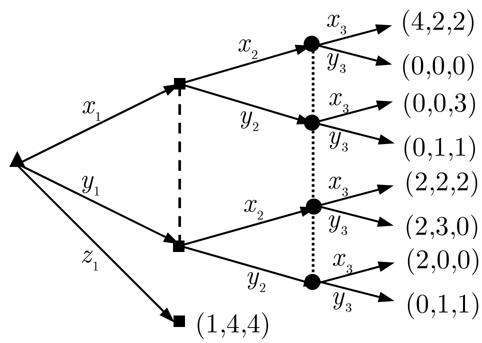

Figure 3.4: Another Example of Sequential Equilibrium

3.6.2 Another Application

There are three players and two pure-strategy N.E. for this game, namely (x_1, x_2, x_3) and (z_1, y_2, y_3).

(Q1) Why is (x_1, x_2, x_3) a N.E.? Because given (x_1, x_2), x_3 is the best response for player 3; given (x_1, x_3), x_2 is the best response for player 2; given (x_2, x_3), x_1 is the best response for player 1. Is (x_1, x_2, x_3) a S.E.? Yes, because the only possible defector in this equilibrium is player 1 whose payoff is at the maximum level when player 1 doesn't defect. The fact that the only possible defector won't defect preserves the equilibrium (x_1, x_2, x_3) as a S.E.

(Q2) Why is (z_1, y_2, y_3) a N.E.? Because given (y_2, y_3), z_1 is the best response for player 1. Is it also a S.E. then? It is a bit tough in this case. We are going to prove that it is not a S.E. as follows.

Suppose that player 1 defects. Player 2 knows that player 1's action is out of equilibrium, but not sure which of x_1 and y_1 was played. Let player 2's belief about the action of player 1 be that it assigns p_2 to action x_1 and $1 - p_2$ to action y_1. We impose a crucial assumption here that there is no correlation in defection. That is to say, player 3 won't defect from the equilibrium as a result of player 1's defection although player 3 knows player 1 defected. Player 2's expected payoff, in response to y_3, is:

$$u_2(x_2; \hat{z}_1, y_3) = p_2 \cdot 0 + (1 - p_2) \cdot 3 = 3 - 3p_2;$$
$$u_2(y_2; \hat{z}_1, y_3) = p_2 \cdot 1 + (1 - p_2) \cdot 1 = 1.$$

Player 2's best response to $(\hat{z}_1, y_3)$ will be to play x_2 if $p_2 \leq \frac{2}{3}$ and to play y_2 otherwise. (Note that here $\hat{z}_1$ means any action other than z_1.)

Moreover, when player 1 defected, player 3 cannot tell whether or not player 2 defected, so player 3 would assume that player 2 doesn't defect by the assumption of no correlation in defection. Let player 3's belief about action of player 1 be that it assigns p_3 to x_1 and $1 - p_3$ to y_1. Then the expected payoff for player 3, in response to y_2, will be:

$$u_3(x_3; \hat{z}_1, y_2) = p_3 \cdot 3 + (1 - p_3) \cdot 0 = 3p_3;$$
$$u_3(y_3; \hat{z}_1, y_2) = p_3 \cdot 1 + (1 - p_3) \cdot 1 = 1.$$

Player 3's best response to $(\hat{z}_1, y_2)$ will be to play x_3 if $p_3 \geq \frac{1}{3}$ and to play y_3 otherwise.

Note that in dealing with Sequential Equilibrium in the case of more than two players, a Sequential Equilibrium would require that the updated beliefs upon defection have to be shared by all non-defectors. In this case, it says that both player 2 and player 3 share the same belief, i.e., the common probability $p(x_1)$ that player 1 played x_1.

(a) Suppose that the common belief is such that $p(x_1) \leq \frac{1}{3}$, then player 2 will choose the best response x_2 and player 3 will choose the best response y_3. In this case, the payoff vector is $(0, 0, 0)$ if player 1 defects with playing x_1 and $(2, 3, 0)$ if player 1 defects with playing y_1. Comparing to the equilibrium payoff $(1, 4, 4)$, player 1 will surely defect with playing y_1.

(b) Suppose that the common belief is such that $\frac{1}{3} < p(x_1) < \frac{2}{3}$, then player 2 will choose the best response x_2 and player 3 will choose the best response x_3. In this case, the payoff vector is $(4, 2, 2)$ if player 1 defects with playing x_1 and $(2, 2, 2)$ if player 1 defects with playing y_1. Comparing to the equilibrium payoff $(1, 4, 4)$, player 1 will surely defect with playing x_1.

(c) Suppose that the common belief is such that $p(x_1) \geq \frac{2}{3}$, then player 2 will choose the best response y_2 and player 3 will choose the best response x_3. In this case, the payoff vector is $(0, 0, 3)$ if player 1 defects with playing x_1 and $(2, 0, 0)$ if player 1 defects with playing y_1. Comparing to the equilibrium payoff $(1, 4, 4)$, player 1 will surely defect with playing y_1.

Concluding from the cases above, we see that regardless of the common belief, player 1 will always defect and thus the equilibrium (z_1, y_2, y_3) is not a S.E.

3.7 Is Sequential Equilibrium Stringent Enough?

While Sequential Equilibrium is a good refinement to Nash Equilibrium, it still leaves room to some nonsensible equilibrium outcomes. Let's take a look at the following game of extensive form, which has two pooling S.E., namely (R, D) and (L, U).

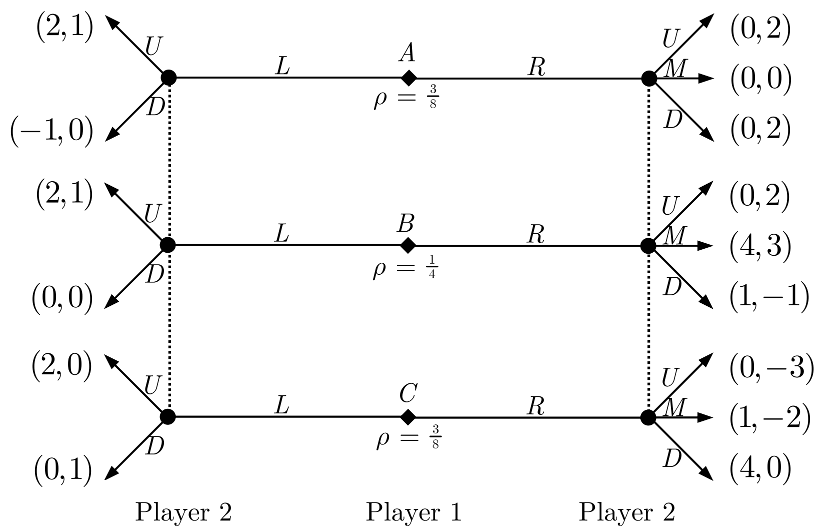

Figure 3.5: Is Sequential Equilibrium Stringent Enough?

(Q1) Why is (R, D) a S.E.? First of all, we need to show that it is a N.E. Since all types of player 1 will play R, player 2's posterior belief about types of player 1 is the same as the priors. The expected payoff for player 2, in response to R, is:

$$
\begin{aligned}
u_2(U; R) &= \tfrac{3}{8} \cdot 2 + \tfrac{1}{4} \cdot 2 + \tfrac{3}{8} \cdot (-3) = \tfrac{1}{8}; \\
u_2(M; R) &= \tfrac{3}{8} \cdot 0 + \tfrac{1}{4} \cdot 3 + \tfrac{3}{8} \cdot (-2) = 0; \\
u_3(D; R) &= \tfrac{3}{8} \cdot 2 + \tfrac{1}{4} \cdot (-1) + \tfrac{3}{8} \cdot 0 = \tfrac{1}{2}.
\end{aligned}
$$

Clearly, player 2's best response will be D, corresponding to player 1's move R. Also note that given player 2's strategy of moving D, player 1's move L is dominated by move R. Hence (R, D) is a N.E.

To show that (R, D) is a S.E., we need to specify beliefs[2] for player 2 in response to player 1's defection. The only defection possibility is that player

[2] Once again, note that when making an equilibrium be a Sequential Equilibrium, we

1 plays L. If we specify that player 2 believes that player 1 who plays L is of type C for sure, i.e. $\mu(\tau_1 = C|L) = 1$, then player 2's best response will be D. As we mentioned before, when player 2's strategy is to play D, player 1's move L is dominated by move R. Therefore, no type 1 will defect and the equilibrium (R, D) is sustained as a S.E.

(Q2) Why is (L, U) a S.E.? Using the similar technique to find out the expected payoff, we can easily verify that (L, U) is a N.E. To show that (L, U) is a S.E., we can specify that $\mu(\tau_1 = A|R) = 1$ and let player 2's move be U. Consequently, all types of player 1 have payoff of 0 from defection instead of 2 at equilibrium. Therefore, (L, U) is a S.E.

But among these two S.E., we claim that (R, D) is not economically sensible. Why? In this equilibrium, type C of player 1 will never defect from its highest payoff choice. Since type C of player 1 won't ever defect, player 2 notices this fact and thus concentrate his beliefs on type A and B. Player 2 will choose U in response[3] to the observance of the out of equilibrium signal L that is believed to be sent from either type A or B. Given player 2's response U, types A and B of player 1 will defect for sure because of higher payoff relative to their respective equilibrium payoffs. To redress this issue and eliminate (R, D) as a sensible equilibrium, we have the next refinement to N.E., i.e., Cho-Kreps' "Intuitive Criterion."

3.8 Cho-Kreps' Intuitive Criterion

A sequential equilibrium fails the *Intuitive Criterion* if there exists an out of equilibrium message m' and a proper subset $J(m')$ of $\overline{T}$ such that:

(1) $\forall \tau \in J(m')$ and $\forall r' \in BR(\overline{T}, m')$, $u^*(\tau) \geq u(\tau, m', r')$; and

(2a) $\exists \tau' \in \overline{T} \backslash J(m')$ s.t. $u^*(\tau') < u(\tau', m', r')\ \forall r' \in BR(\overline{T} \backslash J(m'), m')$.

Condition (1) states that types $\tau \in J(m')$ will never defect under all beliefs regarding all types. Condition (2a) states that for all beliefs regarding the defection candidate group, there exists at least one defection candidate who would be better off by defecting from the equilibrium.

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need to specify only one belief for player 2 in observance of the zero probability event, i.e., out of equilibrium move by player 1, that make player 1 not defect.

[3] Note that once the defection is observed, don't update player 2's beliefs using Bayes' rule because the priors won't matter by now. Although we discard the priors in this case, "Perfect Sequential Equilibrium" would pick up the priors and update player 2's belief anyway.

A sequential equilibrium also fails the *Intuitive Criterion* if condition (1) holds and condition (2a) is replaced by the following condition:

(2b) $\exists \tau' \in \overline{T} \backslash J(m')$ s.t. $u^*(\tau') < \min_{r' \in BR(\overline{T} \backslash J(m'), m')} u(\tau', m', r')$.

This condition (2b) says essentially the same thing as condition (2a). Let's consider one member of the defection candidate group. If the equilibrium payoff for this member is less than the minimum among all payoffs for him under all best responses (i.e., beliefs regarding the defection candidate group), then this defection candidate will end up defecting, and thus the equilibrium fails Intuitive Criterion.

Application Rule: Step 1: find out the types $J(m')$ that will never defect under all beliefs regarding all types. At two polar cases, if it turns out that $J(m') = \emptyset$ then the equilibrium passes the Intuitive Criterion trivially since Step 1 cannot be carried out; and if it turns out that $J(m') = \overline{T}$ then the sequential equilibrium passes the Intuitive Criterion since no type will ever defect and thus it is useless to proceed further. Step 2: if there exists at least one defection candidate who would benefit from defection under all beliefs regarding the defection candidate group, then the sequential equilibrium fails the Intuitive Criterion.

(Q1) As an application, why does the sequential equilibrium (R, D) fail the Intuitive Criterion? Step 1: Let's find out the types of player 1 that will never defect. As we mentioned before, $J(m' = L) = \{C\}$. Step 2: If at least one of the remaining types $\{A, B\}$ will defect under all beliefs, then the sequential equilibrium is said to fail the Intuitive Criterion. In this case, player 2's best response to $m' = L$ is U for $\overline{T} \backslash J(m' = L) = \{A, B\}$ and this response U surely makes both types A and B better-off relative to their respective equilibrium payoffs. Therefore, both types will defect and thus the sequential equilibrium (R, D) fails the Intuitive Criterion.

(Q2) We may also ask ourselves why does the other sequential equilibrium (L, U) pass the Intuitive Criterion? Step 1: Let's find out the types of player 1 that will never defect under this circumstance. If type A defects then he will receive 0 units of payoff regardless of player 2's response, which is worse-off than his equilibrium payoff of 2 units. Both type B and C are hopeful of benefiting from a possible defect, so we have $J(m' = R) = \{A\}$. Step 2: We need to find out the best response set for player 2 for all possible defection candidates, $\overline{T} \backslash J(m' = R) = \{B, C\}$. Denote as $p(B)$ player 2's assessment of the probability that the defector is type B. His assessment of the probability that the defector is type C is $1 - p(B)$. The player 2's expected payoff is then:

$$\begin{aligned} u_2(U;R) &= p(B)\cdot 2 + [1-p(B)]\cdot(-3) = 5p(B)-3; \\ u_2(M;R) &= p(B)\cdot 3 + [1-p(B)]\cdot 2 = 5p(B)-2; \\ u_2(D;R) &= p(B)\cdot(-1) + [1-p(B)]\cdot 0 = -p(B). \end{aligned}$$

Clearly, U is dominated by M in this case and thus the best response set for player 2 is $BR(\overline{T}\backslash J(m'=R), m'=R) = \{M, D\}$. Player 2 will play M if $p(B) > \frac{1}{3}$ and play D otherwise.

Suppose that $p(B) > \frac{1}{3}$ holds, then player 2's best response is to play M, and the payoffs for B and C are 4 and 1, respectively. Suppose that $p(B) < \frac{1}{3}$, then player 2's best response is to play D, and the payoffs for B and C are 1 and 4, respectively. Suppose that $p(B) = \frac{1}{3}$ holds, then player 2's response is $\pi(M|m') = \pi(D|m') = \frac{1}{2}$ and the payoffs for type B and C are both 2.5. For type B, the minimum of his payoff from three possible beliefs from player 2 is 1, which is smaller than his equilibrium payoff level 2. Type B won't defect for all beliefs. Similarly, type C won't defect for all beliefs either. Therefore, the sequential equilibrium (L, U) passes the Intuitive Criterion.

3.9 Strengthened Intuitive Criterion

A sequential equilibrium fails the *Strengthened Intuitive Criterion* (a.k.a. Forward Induction Equilibrium) if condition (1) holds and condition (2a) is replaced by the condition:

(2c) $\forall r \in BR(\overline{T}\backslash J(m'), m'),\ \exists \tau' \in \overline{T}\backslash J(m')$ s.t. $u^*(\tau') < u(\tau', m, r)$.

Condition (2c) is different from condition (2a) in the following sense. When we verify whether or not a sequential equilibrium fails the Intuitive Criterion, we fix the type of all possible defection candidates one at a time and then examine all possible posterior beliefs (i.e., best responses) of the uninformed agent for each type sequentially. When we verify whether a sequential equilibrium fails the Strengthened Intuitive Criterion, we fix instead all possible posterior beliefs one at a time and then we examine each of all possible defection candidates under each specified belief. If for each of all possible beliefs, there exists at least one defection candidate who will defect, then the equilibrium fails the Strengthened Intuitive Criterion; on the other hand, if for at least one possible belief, there is no defection candidate who will defect, then the equilibrium passes the test.

Under the Intuitive Criterion, a defection candidate defects only if he will defect under all possible beliefs. Under the Strengthened Intuitive Criterion, a defection candidate defects if he will defect under one belief. The Strengthened Intuitive Criterion is more stringent in that it is easier to find types that will defect under this criterion and thus we are going to reject more sequential equilibrium than using the Intuitive Criterion.

Now we want to demonstrate that the sequential equilibrium (L, U), which passes the Intuitive Criterion, won't survive the Strengthened Intuitive Criterion. Step 1 is the same as before and we have $\overline{T} \backslash J(m' = R) = \{B, C\}$. Step 2: Since fixing responses is the same as fixing beliefs, we suppose that $p(B) > \frac{1}{3}$ and player 2's best response is M. Given player 2's best response M, is there a defection candidate who will defect? Sure, type B will be happy to, since the defection will bring him 4 units of payoff, comparing to 2 units at equilibrium. Now let's suppose $p(B) < \frac{1}{3}$ and player 2's best response will be D. Given player 2's best response D, is there a defection candidate that will defect? Sure, type C will be happy to, since the defection will bring him 4 units of payoff, comparing to 2 units at equilibrium. Finally let's suppose $p(B) = \frac{1}{3}$ and player 2's best response is $\pi(M|m') = \pi(D|m') = \frac{1}{2}$. Given this best response, is there a defection candidate that will defect? Both type B and C will be happy to, since the defection will bring each of them 2.5 units of payoff, comparing to 2 units at equilibrium. Regardless of player 2's belief, there is always at least one type to defect, so the sequential equilibrium (L, U) fails the Strengthened Intuitive Criterion. Cho-Kreps Intuitive Criterion says that type B won't defect for all the beliefs, and type C won't defect for all the beliefs either. But here type B will defect in one set of belief and type C will defect in another.

Is there anything at all in this game that will pass the Strengthened Intuitive Criterion then? Yes, (L, U) by type A and B and (R, D) by type C. How can we prove this? Because there is no out of equilibrium move at all in this situation. Player 2 will always regard the move of R as initiated by type C, and all moves of L as by types A and B. Moreover, type C cannot gain anything from defection and types A, B cannot gain anything from defection either.

3.10 Divinity; Universal Divinity; Never a Weak Best Response

Let's denote the equilibrium utility for player 1 as

$$u^*(\tau) = \sum\nolimits_{m \in M} u(m;\tau,r) \cdot \phi(m;\tau).$$

The set of best responses that make it optimal for type τ to defect is written as

$$D(\tau|m) = \{r \in MBR(\overline{T}, m) | u^*(\tau) < u(m;\tau,r)\}.$$

The set of best responses that make it indifferent for type τ between defecting and not defecting is written as

$$D^0(\tau|m) = \{r \in MBR(\overline{T}, m) | u^*(\tau) = u(m;\tau,r)\}.$$

D1 Condition: Suppose for τ and $\tau' \neq \tau$, $D(\tau|m) \cup D^0(\tau|m) \subset D(\tau'|m)$, then D1 condition holds.

Let's call the union of the defection-prone best response set and the defection-indifferent best response set for type τ as the best response set for type τ weakly willing to defect. What Condition D1 says is that the best response set for type τ' strictly willing to defect contains that for type τ weakly willing to defect. That is equivalent to say that the belief set for type τ' strictly willing to defect is greater than that for type τ weakly willing to defect. Literally speaking, it says that if type τ is weakly willing to defect then type τ' is strictly willing to defect; type τ' is more likely to defect than type τ.

D1 Equilibrium: If the D1 Condition holds, the posterior assessment is $\mu(\tau|m) = 0$.

In the D1 Equilibrium, the uninformed agent assigns zero probability to the type τ who is less likely to defect, in response to the out-of-equilibrium message m.

Divinity: Suppose the D1 condition holds, then $\frac{\mu(\tau|m)}{\rho(\tau)} \leq \frac{\mu(\tau'|m)}{\rho(\tau')}$.

The Divinity says that if τ' is more likely to defect than τ is to defect, then the uninformed agent cannot assign more weight to type τ in forming the posterior assessment. If we manipulate the inequality into the following equivalent form,

$$\frac{\mu(\tau'|m)}{\mu(\tau'|m) + \mu(\tau|m)} \geq \frac{\rho(\tau')}{\rho(\tau') + \rho(\tau)},$$

then it becomes clear that the relative weight of posterior belief assigned to type τ' must be no less than the relative weight of priors for type τ', in order to reflect the fact that type τ' is more likely to defect. In the case where we have only two types, either τ' or τ, the divinity boils down to $\mu(\tau'|m) \geq \rho(\tau')$, which makes perfect sense.

D2 Condition: Suppose that there exists a type τ such that

$$D(\tau|m) \cup D^0(\tau|m) \subset \bigcup\nolimits_{\tau' \neq \tau} D(\tau'|m),$$

then the D2 criterion is said to hold.

This condition says that the best response set for type τ weakly willing to defect is contained within the best response set for at least one other type $\tau' \neq \tau$ strictly willing to defect, i.e., it is more likely for at least one other type $\tau' \neq \tau$ than type τ strictly willing to defect.

Universal Divinity: If D2 Condition holds, the posterior assessment is $\mu(\tau|m) = 0$.

This is a more stringent equilibrium concept than D1 Equilibrium. D1 Condition may not hold for one particular $\tau' \neq \tau$, but once D2 Condition holds it must be the case that there is at least one type $\tau' \neq \tau$ such that D1 holds. The converse may not be true.

Never a Weak Best Response (NWBR): Suppose that there exists a type τ such that

$$D^0(\tau|m) \subset \bigcup\nolimits_{\tau' \neq \tau} D(\tau'|m),$$

then the posterior assessment is $\mu(\tau|m) = 0$.

Application Rule: Step 1: verify if the respective condition (D1 Condition, D2 Condition, NWBR Condition) holds. If the condition doesn't hold, then the equilibrium passes the respective refinement trivially. Step 2: If the condition holds, assign the posterior assessment according to respective refinement (D1 Equilibrium, Divinity, Universal Divinity, NWBR) to get the best response under the updated beliefs. Next use the best response to evaluate all types concerned, including the type less likely to defect, to see if any type will defect. If there exists at least one type that will defect (it doesn't matter which particular type will defect), given the posterior beliefs assigned, then it is said to fail the respective equilibrium concept.

Note the following nesting relationship between the various refinements of the Nash Equilibrium. Strategic Stability Equilibrium $\subseteq$ NWBR $\subseteq$ Universal Divinity $\subseteq$ D1 Equilibrium $\subseteq$ Strengthened Intuitive Criterion $\subseteq$ Cho-Kreps' Intuitive Criterion $\subseteq$ Sequential Equilibrium $\subseteq$ Subgame Perfection

Nash Equilibrium $\subseteq$ Nash Equilibrium. Moreover, we know that Perfectly Sequential Equilibrium $\subseteq$ Cho-Kreps' Intuitive Criterion.

3.11 A Few Applications

3.11.1 Spence Model Revisited

Let's start with the Spence's model where the zero profit wage rate is $n \times e$, and $n = 1$ for low ability worker and $n = 2$ for high ability worker. Recall that e is the signal the informed agent sent to the employer, education level. The low-type worker's optimization problem is $\max_e e - k_1 e^2$ and has a first-best solution e_1^*.

The high-type worker's optimization problem is

$$\max_e 2e - k_2 \cdot e^2 \text{ s.t. } 2e - k_1 \cdot e^2 \leq e_1^* - k_1 \cdot e_1^{*2}.$$

The incentive compatibility condition $2e - k_1 \cdot e^2 \leq e_1^* - k_1 \cdot e_1^{*2}$ ensures that the low-type worker will not mimic the high-type worker's education level. We consider at first possible pooling equilibrium and let the uninformed agent have the following belief $\mu(\tau_p|e_p)$ about the concentration of high-ability worker among the workers' pool such that $w(e_p, \mu) < 2e_p$, where e_p is the pooling education level.

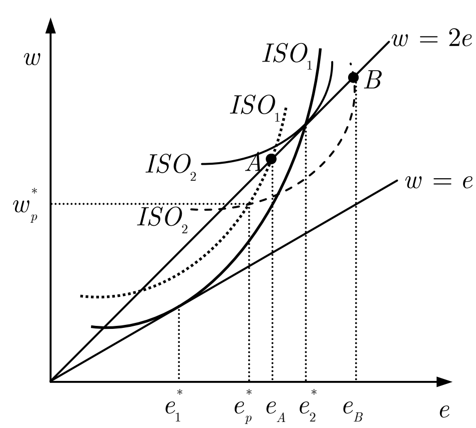

Figure 3.6: Spence Model with Two Types of Workers

As usual, we assume that low-type worker has a steeper iso-utility curve. In the graph above, utility levels are getting higher towards northwest, and profit levels are getter higher towards southeast. The low-type worker will occupy his first-best solution with education level e_1^* and his iso-utility curve ISO_1 cuts the high-type wage schedule at the point where the high-type worker will signal education level of e_2^*. (e_1^*, e_2^*) is the separating equilibrium as we discussed before. Note that the exact location of (e_p^*, w_p^*) will depend upon the uninformed agent's belief $\mu(\tau_p|e_p)$.

(Q1) In the graph above, why is (e_p^*, w_p^*) part of a Sequential Equilibrium?

Let's specify the out of equilibrium belief for the uninformed to be $\mu(\tau_1|e \neq e_p^*) = 1$. Then the prevailing wage schedule is $w = e$ for any defector, which will make both types worse off upon defection. Therefore, the pooling equilibrium is sustained as a Sequential Equilibrium.

(Q2) Is (e_p^*, w_p^*) a part of Cho-Kreps Equilibria?

Step 1: find out the never-defect candidates group. Since the ISO_1 through the pooling equilibrium cuts the high-type wage at point A, any education level higher than e_A will make the low-type worse-off comparing to the pooling equilibrium. Since the ISO_2 through the pooling equilibrium cuts the high-type wage at point B, there is a portion of education levels lower than e_B will make the high-type better-off comparing to the pooling equilibrium. So for any out of equilibrium signal $e' \in (e_A, e_B)$, the low-type will never defect and thus $J(e') = \{1\}$. Step 2: find out the best response set by the uninformed agent. Given that the low-type will never defect, the uninformed agent forms the belief that $\mu(\tau_2|e') = 1$ and thus gives the response $w(e') = 2e'$. Therefore, the high-type worker will defect for sure and thus the pooling equilibrium fails the Cho-Kreps Intuitive Criterion. As a matter of fact, any pooling equilibrium in a two-type signaling model will never be Cho-Kreps Equilibrium.

(Q3) Is the separating equilibrium (e_1^*, e_2^*) a part of Cho-Kreps Equilibria?

First note that the only possible defection comes from a pooling equilibrium. Suppose that there is one type will defect into a pooling equilibrium, then the immediate consequence is that the other type will defect as well, by the definition of a pooling equilibrium. Therefore, we cannot find any education level e' such that only one type would defect, i.e., $J(e') = \emptyset$. So we cannot apply Step 1 for Cho-Kreps and the separating equilibrium fails the Cho-Kreps Intuitive Criterion trivially. The intuition behind is that as long as the uninformed agent's posterior belief is weighed sufficiently in favor of the high-type worker, then both types will defect.

3.11.2 Spence Model with three types of workers

Next, we work on an extension of Spence model into three-type workers, with ability levels $n = 1, 2, 3$. The graph is shown below.

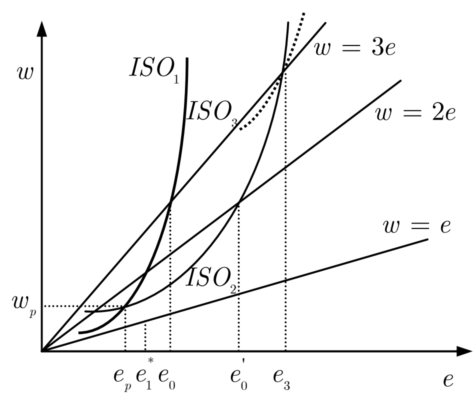

Figure 3.7: Spence Model with Three Types of Workers

(Q4) Is the partial pooling equilibrium (e_p, e_3) part of a Cho-Kreps Equilibrium?

It is easy to show that (e_p, e_3) is a Sequential Equilibrium, for example, by specifying the posterior belief in reaction to the out of equilibrium education level $e' \neq e_p, e_3$ as $\mu(\tau_1|e') = 1$. In the graph above, we set up $e_p < e_1^*$ so that type 1 will not defect upon correctly identified as type 1. Clearly if type 2 defects with $e' > e_0$[4], then type 1 won't follow type 2's education level. So for type 2's out of equilibrium move $e' \in (e_0, e_0')$, we have $J(e') = \{1\}$ and thus the uninformed agent concentrates the belief on type 2 and 3. As a consequence, the best response must be $w \geq 2e'$, and thus type 2 would defect for sure. Therefore, the partial pooling equilibrium (e_p, e_3) fails Cho-Kreps Intuitive Criterion. Also note that for type 2's out

[4] Why would we choose e_0 corresponding to the intersection between ISO_1 and $w = 3e$, not the intersection between ISO_1 and $w = 2e$? Because in Step 1 for Cho-Kreps we need to find out a type that won't defect for any posterior beliefs. Should we choose the latter intersection point, the belief $\mu(\tau_3|e') = 1$ may make τ_1 defect.

of equilibrium move $e' > e'_0$, it may be the case that type 2 will not defect for some beliefs.

Next, let's draw the graph for the three-type signaling game a bit differently.

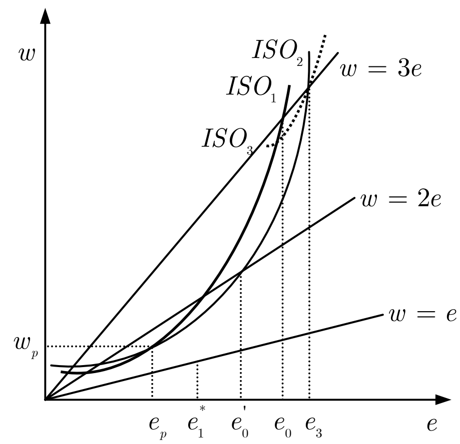

Figure 3.8: Spence Model in presence of Almost Indistinguishable Types

(Q5) Is the partial pooling equilibrium (e_p, e_3) part of Cho-Kreps' Equilibria?

As we demonstrated in (Q4), it is easy to show (e_p, e_3) is a Sequential Equilibrium. Once again, we have $e_p < e_1^*$ so that type 1 will not defect upon correct identification as type 1. If type 2 defects with $e' > e_0$, then type 1 won't follow type 2's move. However, for type 2's out of equilibrium move $e' > e_0$, we are not sure whether type 2 will defect in the end. If the posterior belief is weighed sufficiently in favor of type 2, then type 2 could be worse off if adopting the defection move (because we don't have $e'_0 > e_0$ in this graph). The fact that type 2 will not defect under this belief implies that type 3 will not defect either (because otherwise the defected type 3 would receive a wage schedule very close to $w = 2e$.). In fact, if the belief is heavily skewed in favor of type 2, we could have $J(e') = \{1, 2, 3\} = \overline{T}$. Because we cannot find at least one type that will defect under all beliefs, the partial pooling equilibrium passes Cho-Kreps Intuitive Criterion.

We may wonder what determines a partial pooling equilibrium that survives Cho-Kreps Intuitive Criterion? Equivalently speaking, what determines $e'_0 < e_0$ in this case? There are two possible conditions. (a) If type

2 is very similar to type 1 so that the slope of ISO_2 is sufficiently close to ISO_1 and the two iso-utility curves are very close to each other; or (b) the wage schedule for type 3 becomes much steeper so that the never-mimicking education level e_0 for type 1 becomes much higher. The intuition behind condition (a) is that when type 1 and 2 are becoming sufficiently close, then neither of type 1 or 2 will defect from the partial pooling equilibrium and thus (e_p, e_3) is part of Cho-Kreps Equilibrium. (Once we know type 1 and 2 will never defect, we know for sure that type 3 will not defect under some beliefs and thus Step 2 won't satisfy.) The intuition behind condition (b) is that the wage schedule for type 3 is so high that the out of equilibrium move $e' > e_0$ must be sufficiently high so that type 1 will not be attracted to mimic. The fact that $e' > e_0$ lies in such a high level drives type 2 not to defect. Once again, the fact that neither type 1 nor type 2 will defect implies that type 3 cannot defect under all beliefs, and thus (e_p, e_3) is part of Cho-Kreps Equilibria.

We now are going to show that a pooling equilibrium never survives Universal Divinity. Note that this conclusion is true in only signaling games where the informed agent moves first. We use a three-type signaling game (in the immediately above graph) for illustration purpose. Let the utility of type i be $w_i - k_i \cdot e^2$, where $w_i = i \cdot e, i = 1, 2, 3$. As usual we assume that type 1 has lower ability and $k_1 > k_2$.

(Q6) Does the partial pooling equilibrium (e_p, e_3) survive Universal Divinity?

Denote as $w^i_{\min}$ the minimum wage needed to induce type i to defect with $e > e_p$. For type 1, we have $w^1_{\min} - k_1 \cdot e^2 = w_p - k_1 \cdot e_p^2$ and thus $w^1_{\min} = w_p + k_1 \cdot (e^2 - e_p^2)$. To induce defection by type 2, we have $w^2_{\min} - k_2 \cdot e^2 = w_p - k_2 \cdot e_p^2$ and thus $w^2_{\min} = w_p + k_2 \cdot (e^2 - e_p^2)$. Since $k_1 > k_2$, we have $w^1_{\min} > w^2_{\min}$. Using the notations in Universal Divinity, we get:

$$D^0(\tau_1, e) = w^1_{\min}(e); D(\tau_1, e) = (w^1_{\min}(e), 3e];$$
$$D^0(\tau_2, e) = w^2_{\min}(e); D(\tau_2, e) = (w^2_{\min}(e), 3e].$$

Clearly, the following nesting relationship holds,

$$D^0(\tau_1, e) \cup D(\tau_1, e) = [w^1_{\min}(e), 3e] \subset (w^2_{\min}(e), 3e] = D(\tau_2, e).$$

That is, type τ_2 is more likely than type τ_1 to defect and thus the posterior belief is $\mu(\tau_1|e > e_p) = 0$. If we put this belief in the context of the graph immediately above, then type 2 will defect with $e \in (e_p, e_0')$ under any best response and the partial pooling equilibrium won't survive Universal Divinity. In essence, the type 2 will guess about the uninformed agent's belief about the defector's type and make decision whether or not to defect accordingly.

One difference between Cho-Kreps and Universal Divinity is that under Cho-Kreps for any $e > e_p$, it is possible that the defector is mistakenly believed as type 1 so that type 2 has to defect with $e > e_0$, a higher education level than e_0'. Yet under Universal Divinity, the posterior belief is $\mu(\tau_1|e > e_p) = 0$ so that no such mis-belief exists.

3.11.3 NWBR vs. Universal Divinity

In the game below, we are going to show that NWBR is even stronger than Universal Divinity.

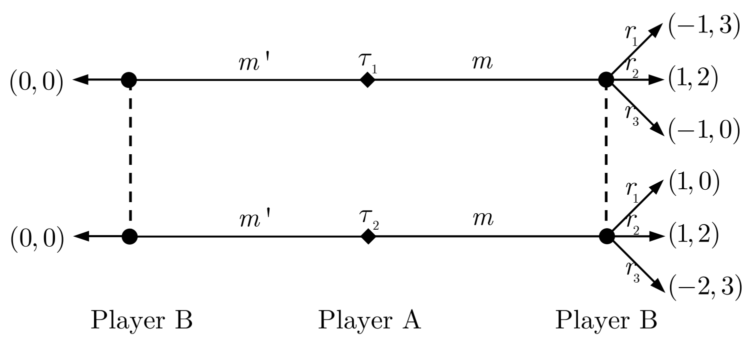

Figure 3.9: Never a Weak Best Response vs. Universal Divinity

(Q7) Is the equilibrium (0,0) Universal Divinity?

Clearly, we can specify the posterior belief for player B as $\mu(\tau_2|m) = 1$. If player A sends the out of equilibrium signal m, the best response by player B will be r_3 under this posterior belief, and thus player A will not defect in the first place. That is, the equilibrium (0,0) is a Sequential Equilibrium. To determine whether (0,0) is Universal Divinity, we need to find out the best response set for player B under all beliefs. Since $v(r_1) = 3 \cdot \rho(\tau_1); v(r_2) =$

$2; v(r_3) = 3 \cdot (1 - \rho(\tau_1))$, we know the best response set is

$$MBR(\mu(\tau_1), m) = MBR(\rho(\tau_1), m) = \begin{cases} r_1 & \text{if } \rho(\tau_1) > \frac{2}{3}; \\ r_2 & \text{if } \frac{1}{3} < \rho(\tau_1) < \frac{2}{3}; \\ r_3 & \text{if } \rho(\tau_1) < \frac{1}{3}; \\ \tilde{r}_{1,2} & \text{if } \rho(\tau_1) = \frac{2}{3}; \\ \tilde{r}_{2,3} & \text{if } \rho(\tau_1) = \frac{1}{3}. \end{cases}$$

Note that when $\rho(\tau_1) = \frac{1}{2}$, we have $v(r_1) = v(r_3) = \frac{3}{2}$. Why don't we use $\tilde{r}_{1,3}$ in this case? Because it is within the range $\frac{1}{3} < \rho(\tau_1) < \frac{2}{3}$ and r_2 dominates $\tilde{r}_{1,3}$ in that $v(r_2) = 2 > \frac{3}{2} = v(\tilde{r}_{1,3})$.

Now we can find out $D(\tau_i|m), D^0(\tau_i|m), i = 1, 2$. Since we know that τ_1 wont' defect for responses r_1 and r_3, we have:

$$\begin{aligned} D(\tau_1; m, r_2) &= \{\pi(r_1) = 0, \pi(r_2) = 1, \pi(r_3) = 0\}; \\ D(\tau_1; m, \tilde{r}_{1,2}) &= \{-\pi(r_1) + [1 - \pi(r_1)] > 0, \pi(r_3) = 0\} \\ &= \left\{0 < \pi(r_1) < \tfrac{1}{2}, \tfrac{1}{2} < \pi(r_2) < 1, \pi(r_3) = 0\right\}; \\ D(\tau_1; m, \tilde{r}_{2,3}) &= \{\pi(r_1) = 0, \pi(r_2) - [1 - \pi(r_2)] > 0\} \\ &= \left\{\pi(r_1) = 0, \tfrac{1}{2} < \pi(r_2) < 1, 0 < \pi(r_3) < \tfrac{1}{2}\right\}; \\ D(\tau_1|m) &= \left\{ \begin{array}{l} \{0 \le \pi(r_1) < \frac{1}{2}, \frac{1}{2} < \pi(r_2) \le 1, \pi(r_3) = 0\} \\ \{\pi(r_1) = 0, \frac{1}{2} < \pi(r_2) \le 1, 0 \le \pi(r_3) < \frac{1}{2}\} \end{array} \right\}; \\ D^0(\tau_1|m) &= \left\{ \begin{array}{l} \{\pi(r_1) = \pi(r_2) = \frac{1}{2}, \pi(r_3) = 0\} \\ \{\pi(r_1) = 0, \pi(r_2) = \pi(r_3) = \frac{1}{2}\} \end{array} \right\}. \end{aligned}$$

In order to apply D1 Condition, we find

$$D(\tau_1|m) \cup D^0(\tau_1|m) = \left\{ \begin{array}{l} \{0 \le \pi(r_1) \le \frac{1}{2}, \frac{1}{2} \le \pi(r_2) \le 1, \pi(r_3) = 0\} \\ \{\pi(r_1) = 0, \frac{1}{2} \le \pi(r_2) \le 1, 0 \le \pi(r_3) \le \frac{1}{2}\} \end{array} \right\};$$

and

$$D(\tau_2|m) \cup D^0(\tau_2|m) = \left\{ \begin{array}{l} \{\pi(r_1) \in [0, 1], \pi(r_2) = 1 - \pi(r_1), \pi(r_3) = 0\} \\ \{\pi(r_1) = 0, \frac{2}{3} \le \pi(r_2) \le 1, 0 \le \pi(r_3) \le \frac{1}{3}\} \end{array} \right\}.$$

Since there is no clear nesting relationship between $D(\tau_1|m) \cup D^0(\tau_1|m)$ and $D(\tau_2|m)$, nor between $D(\tau_2|m) \cup D^0(\tau_2|m)$ and $D(\tau_1|m)$, we cannot remove any defection candidate by assigning a zero posterior belief to the less likely defection candidate. Therefore, the Sequential Equilibrium (0,0) survives Universal Divinity[5]. Our next natural question is whether this Universal Divinity survives NWBR.

[5] As a general note, when we are dealing with two types in a signaling game, D1 Equilibrium is the same as Divinity and Universal Divinity.

(Q8) Does the Universal Divinity (0,0) qualify to be Never a Weak Best Response?

To use NWBR in this two-type model, we need to compare $D^0(\tau_i|m)$ to $D(\tau_j|m)$ where $i, j = 1, 2$ and $i \neq j$. We don't find any nesting relationship between $D^0(\tau_1|m)$ and $D(\tau_2|m)$, but we do find that $D^0(\tau_2|m) \subset D(\tau_1|m)$, which says that type τ_1 is more likely to defect so that we assign the following posterior belief $\mu(\tau_1|m) = 1$ and remove type τ_2 as a defection candidate. Given this posterior belief, player B's best response is r_1. Under this best response from player B, however, type τ_1 will not defect at all since $u(m; \tau_1, r_1) = -1 < 0$ and type τ_2 will defect since $u(m; \tau_2, r_1) = 1 > 0$. Since under this belief and best response for player B, there exists at least one type[6] who will defect, the equilibrium (0,0) is not NWBR.

Does this reasoning make sense? The posterior belief is that type τ_1 will defect and type τ_2 will not defect. But the eventual outcome is exactly the opposite, type τ_1 will not defect and type τ_2 will defect. Note that Universal Divinity has the same problem. An effort to restrain such nonsensible results from occurring is the rationale behind Perfect Sequential Equilibrium, which we will cover next.

3.12 Perfect Sequential Equilibrium

Before we introduce the concept of Perfect Sequential Equilibrium, let's visit some useful theorems.

Definition: *Monotonic Signaling Games*

For $m \in M$ and $\pi(r|m), \pi'(r|m) \in MBR(\overline{T}, m)$, if for some $\hat{\tau} \in \overline{T}$ such that $\sum_{r \in R(m)} u(m; \hat{\tau}, r) \cdot \pi(r|m) > \sum_{r \in R(m)} u(m; \hat{\tau}, r) \cdot \pi'(r|m)$, then for all $\tau \in \overline{T}$, we have $\sum_{r \in R(m)} u(m; \tau, r) \cdot \pi(r|m) > \sum_{r \in R(m)} u(m; \hat{\tau}, r) \cdot \pi'(r|m)$.

Note that most signaling games are monotonic signaling games. Suppose that the response r is a monetary payment, then it is assumed that the informed agent prefer more money to less, regardless of m and τ. That is, for any out of equilibrium message, every type $\tau \in \overline{T}$ prefers more money to less.

[6] Note a very crucial point here is that neither Universal Divinity nor NWBR specifies which particular type would defect from the equilibrium. All they say is that if at least one type will defect after all criteria are satisfied, then the equilibrium fails Universal Divinity or NWBR.

Theorem 1: In genetic monotonic signaling games, Condition D1 is equivalent to Universal Divinity and NWBR and Strategic Stability. [A result from Cho-Sobel working paper (1988)]

Theorem 2: (*Uniqueness*) Suppose the following conditions hold,

(A1) if $r > r'$, then $u(m;\tau,r) > u(m;\tau,r')$ for the informed agent;

(A2) $v(m;\tau,r)$ for the uninformed agent is continuous in all its arguments and is a strictly quasi-concave differentiable function of r;

(A3) $\partial v/\partial r$ is increasing in τ;

(A4) (*Single-Crossing Property*) $u(m;\tau,r)$ is differentiable in m and r, and $-\frac{\partial u/\partial m_i}{\partial u/\partial r}$ is decreasing in $\tau, \forall i \in \{1,2,...,N\}$;

then the unique equilibrium surviving Condition D1 is also a Riley Reactive Equilibrium.

One note about the relation to Riley Reactive Equilibrium is that when the single-crossing property holds, we can always show nesting relationship among best response sets for different types. Now we are ready to discuss the Perfect Sequential Equilibrium.

Suppose that there exists a set $\hat{K}$ such that $\hat{K} = \{\tau \in \overline{T} | u^*(\tau) \geq \max_{r\in R(m)} u(m;\tau,r)\}$, i.e., the set $\hat{K}$ is the types weakly unwilling to defect. Define $K \equiv \overline{T} \backslash \hat{K}$. Then player B, the uninformed agent, should concentrate the posterior beliefs on the defection candidates $K \subseteq \overline{T}$,[7] and in particular, using Bayes rule to update the beliefs as conditional distribution over types who are strictly willing to defect. For $K \subseteq \overline{T}$ such that the priors $\rho(k \in K) > 0$ and types $\tau \in K$, the posterior beliefs in reaction to out of equilibrium message m, are formed as

$$\mu(\tau|m) = \frac{\rho(\tau)}{\sum_{k\in K}\rho(k)}, \forall \tau \in K.$$

Given the out of equilibrium message m and posterior beliefs $\mu(\tau|m)$, let the response r be a member of the mixed-best response set for player B, i.e., $r \in MBR(\mu(\tau|m), m)$, then a Perfect Sequential Equilibrium requires that

$$\{\tau \in \overline{T} | u^*(\tau) \leq u(m;\tau,r)\} = K,$$

[7] As we have mentioned above about the rationale behind the Perfect Sequential Equilibrium, the next steps will be to compute the best response sets and make sure that the types who won't defect are in the set $\hat{K}$. This is more or less a fixed-point requirement.

and

$$\{\tau \in \overline{T} | u^*(\tau) \geq u(m; \tau, r)\} = \hat{K}.$$

That is, the eventual results are consistent with the posterior beliefs. If $K \neq \emptyset$, then the equilibrium is said to fail the Perfect Sequential Equilibrium.

Note that up to now we are saying that a type τ will ***not*** defect if the out of equilibrium message m subject to the best response associated with the posterior beliefs produces the same level of utility as that at equilibrium, i.e., $u^*(\tau) = u(m; \tau, r)$. But here we are essentially saying that when $u^*(\tau) = u(m; \tau, r)$ holds, then the type τ defects with some positive probability so that both $\{\tau \in \overline{T} | u^*(\tau) \leq u(m; \tau, r)\} = K$ and $\{\tau \in \overline{T} | u^*(\tau) \geq u(m; \tau, r)\} = \hat{K}$ hold.

Formally, let $K^s \subseteq K$ be the set of types strictly willing to defect and $K^w \subseteq K$ the set of types weakly willing to defect. Let $h(\tau)$ be the probability that type τ makes the out of equilibrium move m, and

$$h(\tau) = \begin{cases} 1 & \text{if } \tau \in K^s; \\ [0,1] & \text{if } \tau \in K^w; \\ 0 & \text{if } \tau \notin K. \end{cases}$$

Moreover, we have $\sum_{\tau \in K} h(\tau) > 0$ and $K^w \cup K^s = K$.

Let $c(\tau)$ be the posterior probability that type $\tau \in K^s$ made the move m for sure and type $\tau \in K^w$ made the move with probability $h(\tau)$. Then the posterior beliefs are updated as

$$c(\tau) = \begin{cases} \frac{\rho(\tau) \cdot h(\tau)}{\sum_{\tau \in K} \rho(\tau) \cdot h(\tau)} & \text{if } \tau \in K; \\ 0 & \text{if } \tau \notin K. \end{cases}$$

An equilibrium doesn't survive Perfect Sequential Equilibrium if there exists $K \subseteq T$ with $\rho(k \in K) > 0$, a response $r \in MBR(\mu(\tau|m), m)$ and a probability $h(\tau)$ defined above such that $c(\tau) = \mu(\tau), \forall t \in \overline{T}$. That is to say, only if $K = \emptyset$, does the equilibrium pass the Perfect Sequential Equilibrium.

3.13 An Application of Perfect Sequential Equilibrium

In the following signaling game of extensive form, we focus on the equilibrium (0,0).

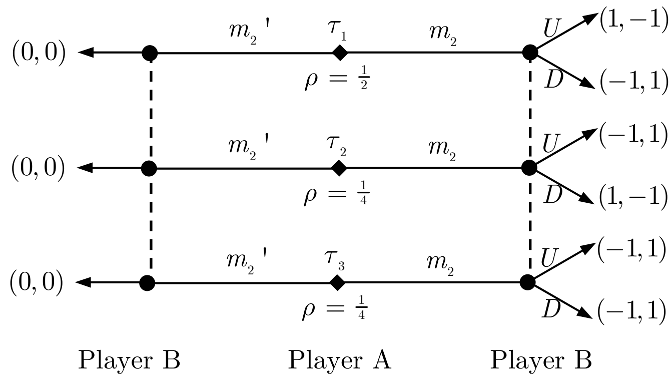

Figure 3.10: One Example of Perfect Sequential Equilibrium

(Q1) Is the equilibrium (0,0) a part of Perfect Sequential Equilibrium?

This equilibrium is a Sequential Equilibrium if we specify the posterior belief in response to the out of equilibrium signal m_2 as $\mu(\tau_3|m_2) = 1$. Under this belief, the best response for player B is $\pi(U|m_2) = \pi(D|m_2) = \frac{1}{2}$. Hence type τ_3 is strictly worse-off upon defection and the other two types are indifferent.

To verify whether or not the equilibrium (0,0) is a part of Perfect Sequential Equilibria, we need to exhaust all possible choices of K, the set of defection candidates. In observance that the out of equilibrium payoff for type τ_3 is always worse than the equilibrium outcome, we know $\tau_3 \notin K$.

(a) Suppose that $K = \{\tau_1, \tau_2\}$ and $\hat{K} = \{\tau_3\}$. The posterior belief is updated, with concentration on types τ_1 and τ_2, as follows,

$$\mu(\tau_1|m_2) = \frac{\rho(\tau_1)}{\rho(\tau_1) + \rho(\tau_2)} = \tfrac{2}{3},$$

and thus $\mu(\tau_2|m_2) = \frac{1}{3}$.Then player B's best response is $\pi(D|m_2) = 1$ and $\pi(U|m_2) = 0$. Therefore type τ_2 defects and type τ_1 won't. We cannot break the equilibrium because the consistency requirement is violated; the outcome $K = \{\tau_2\}$ is not the same as before, $K = \{\tau_1, \tau_2\}$.

(b) Suppose that $K = \{\tau_1\}$ and $\hat{K} = \{\tau_2, \tau_3\}$. The posterior belief will be $\mu(\tau_1|m_2) = 1$ and player B's best response is $\pi(D|m_2) = 1$ and $\pi(U|m_2) = 0$. Therefore type τ_2 defects but type τ_1 won't. Once again, we cannot break the equilibrium because the consistency requirement is violated; the outcome $K = \{\tau_2\}$ is not the same as before, $K = \{\tau_1\}$.

(c) Suppose that $K = \{\tau_2\}$ and $\hat{K} = \{\tau_1, \tau_3\}$. The posterior belief will be $\mu(\tau_2|m_2) = 1$ and player B's best response is $\pi(U|m_2) = 1$ and $\pi(D|m_2) = 0$. Therefore type τ_1 defects and type τ_2 won't. Once again, we now have $K = \{\tau_1\}$, not as before $K = \{\tau_2\}$. We cannot break the equilibrium.

Combining all three sensible cases for K above, we conclude that the equilibrium (0,0) is a Perfect Sequential Equilibrium.

(Q2) Does the equilibrium (0,0) pass the Strengthened Intuitive Criterion?

Once again, we recognize that τ_3 will never defect under any beliefs, and both τ_1 and τ_2 are hopeful of benefiting from defecting under certain beliefs. We have $J(m_2) = \{\tau_3\}$ and $\overline{T}\backslash J(m_2) = \{\tau_1, \tau_2\}$. Let the belief is that the defector is type τ_1 with probability $p(\tau_1)$. The expected payoffs for player B are then $v(U, m_2) = 1 - 2p(\tau_1)$ and $v(D, m_2) = 2p(\tau_1) - 1$. The best response set will be

$$MBR(m_2) = \begin{cases} \pi(U|m_2) = 1 & \text{if } p(\tau_1) < \frac{1}{2}; \\ \pi(D|m_2) = 1 & \text{if } p(\tau_1) > \frac{1}{2}; \\ \pi(U|m_2) = \pi(D|m_2) = \frac{1}{2} & \text{if } p(\tau_1) = \frac{1}{2}. \end{cases}$$

It is now clear that for the belief of $p(\tau_1) < \frac{1}{2}$, τ_1 will defect and τ_2 won't; for the belief of $p(\tau_1) > \frac{1}{2}$, τ_2 will defect and τ_1 won't; for the belief of $p(\tau_1) = \frac{1}{2}$, both τ_1 and τ_2 are indifferent between defecting and staying at the equilibrium. By our standard assumption in the context of Intuitive Criterion and Strengthened Intuitive Criterion, indifference implies no defection. Since under at least one belief, $p(\tau_1) = \frac{1}{2}$, none of the types $\overline{T}\backslash J(m_2) = \{\tau_1, \tau_2\}$ will defect, we conclude that the equilibrium (0,0) passes the Strengthened Intuitive Criterion.

In terms of nesting relationship concerning Perfect Sequential Equilibrium, all we can say is that Perfect Sequential Equilibrium $\subseteq$ Cho-Kreps Intuitive Criterion, and no other nesting relationship is definite.

3.14 References

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Chapter 4

Agency Theory

4.1 Principal-Agent Model

A principal-agent model is also known as a hidden action/technology model. In this context, a principal tries to set up an optimal contract to hire an agent. There is symmetric pre-contract information. The agent chooses action $a \in A$, which is not contractible[1], and produces output $x \in X \equiv [\underline{x}, \overline{x}]$. The precommitted wage is $w(x)$. In general, the output level is determined by $x(\tilde{\theta}, a)$, where $\tilde{\theta}$ is a noise term. In this model, we use an explicit form as $x = a + \tilde{\theta}$. We assume that $E[x_a(\tilde{\theta}, a)] > 0$ holds, i.e., higher effort will produce higher output on average. Note that we don't assume that higher effort will be associated with higher output in each effort level, because the agent won't necessarily get higher wage for higher effort.

The risk-averse agent's utility function is $u(w(x)) - \psi(a)$, where $\psi(a)$ is effort disutility and the reservation utility level is u. We assume further that $u'(\cdot) > 0, u''(\cdot) < 0$, $\psi'(\cdot) > 0$ and $\psi''(\cdot) > 0$. The risk-averse principal's utility function is $v(x - w(x))$ with $v'(\cdot) > 0, v''(\cdot) < 0$. When dealing with the randomness in the model, it has been shown that it is more convenient to work with x directly rather than $\tilde{\theta}$. So we assume that the output has a density of $f(x; a)$ with c.d.f. $F(x; a)$. Furthermore, we assume that output from higher effort first-order stochastic dominates output from lower effort, i.e., $F_a(x; a) < 0, \forall x$. To avoid corner solution, we assume that $F_a(x; a) = F_a(\overline{x}; a) = 0, \forall a$.

[1] In this type of model, we can use "non-observable" and "non-contractible" interchangeably, but we won't have this freedom when dealing with reputation models.

Let's work out the first-best solution assuming that the effort level a is observable. The principal's optimization problem is

$$\max_{w(\cdot),a} \int_{\underline{x}}^{\overline{x}} v(x - w(x))f(x;a)dx \text{ s.t. } \int_{\underline{x}}^{\overline{x}} u(w(x))f(x;a)dx - \psi(a) \geq u.$$

The constraint above is the individual rationality (IR) constraint, a.k.a. participation constraint. We don't need the incentive compatibility (IC) constraint when working with first-best solution. The Lagrangian is setup as

$$\mathcal{L} = \int_{\underline{x}}^{\overline{x}} [v(x - w(x)) + \lambda u(w(x))]f(x;a)dx - \lambda\psi(a) - \lambda u,$$

where λ is the shadow price of income to the agent in each state. Using standard optimal control theory, we get the following two sets of first-order conditions:
(1) w.r.t. $w(x;a)$,

$$\frac{v'(x - w(x))}{u'(w(x))} = \lambda, \forall x \in X;$$

and
(2) w.r.t. a,

$$\int_{\underline{x}}^{\overline{x}} [v(x - w(x)) + \lambda \cdot u(w(x))] \cdot f_a(x;a)dx = \lambda \cdot \psi'(a).$$

The optimal wage contract, called "*forcing contract*," will be in the following form,

$$w(x;a) = \begin{cases} w^*(x) \text{if } a = a^*; \\ 0 \text{otherwise.} \end{cases}$$

The condition (1) is also known as "*Borch Rule*," saying that the ratios of marginal utilities of income are equated across states in an optimal insurance contract.

Note that the above results are for the general case where both the principal and the agent are risk-averse. Suppose that the principal is risk-neutral and the agent is risk-averse, then the first set of first-order-conditions above become

$$\frac{1}{u'(w(x))} = \lambda, \forall x \in X,$$

which implies $w(x) = c$, a flat wage schedule. The intuition behind the flat wage contract is that the principal will bear all the risk since the principal is risk-neutral and the agent is risk-averse. Moreover, if the principal precommits the flat wage to the agent, then the agent will not put as much effort as required by the first-best solution; the precommittment is efficient ex ante but inefficient ex post. The loss of efficiency comes from the deviation from the first-best solution.

As a simple example, suppose that the utility function for the agent takes the form $u(w(x)) = 2\sqrt{w(x)}$ and $\psi(a) = a$. The principal is assumed to be risk-neutral so that $v(x - w(x)) = x - w(x)$. The output x follows an exponential distribution between 0 and ∞; $x \sim \exp(-x/a) \cdot I(x \in (0, \infty))$, where $I(x \in (0, \infty))$ is an indicator function. Clearly, the mean output is a. Let the agent's reservation utility level be $\frac{1}{16}$.

The optimal conditions with respect to $w(x; a)$ say that

$$\frac{1}{u'(w(x))} = \lambda, \text{i.e., } w(x) = \lambda^2.$$

Here we see a flat wage contract as mentioned before. The IR constraint will be binding and the agent will earn reservation utility at the equilibrium, i.e. $2\sqrt{w(x)} - \psi(a) = \frac{1}{16}$, or, $\lambda = \frac{1}{2}(a + \frac{1}{16})$.

The principal's optimization problem then becomes

$$\max_a E(x; a) - E(w(x)) = \max_a a - \lambda^2,$$

which has a solution of $a^* = \frac{31}{16}$. Hence, we have $\lambda = 1$ and $w(x) = 1$. The optimal contract will be

$$w = \begin{cases} 1 & \text{if } a = \frac{31}{16}; \\ 0 & \text{otherwise.} \end{cases}$$

Now let's go back to the general model and deal with the hidden action case, or second-best solution. Since only the agent knows his effort level, the principal solves the following problem,

$$\max_{w(\cdot), a} \int_{\underline{x}}^{\overline{x}} v(x - w(x)) f(x, a) dx \text{ s.t. } \int_{\underline{x}}^{\overline{x}} u(w(x)) f(x, a) dx - \psi(a) \geq u \text{ and}$$
$$a \in \arg\max_{a' \in A} \int_{\underline{x}}^{\overline{x}} u(w(x)) f(x, a') dx - \psi(a').$$

Note that the second constraint here is the Incentive Compatibility constraint. Basically, the principal solves first the IC problem to get the agent's best response set to each wage schedule, $a(w(x))$, then plugs this best response into the principal's utility maximizing problem upon the agent's participation, and finally solves for the optimal wage contract. If there are more than one solution in the IC problem, then the principal chooses the action a that maximizes his own utility.

Although this procedure sounds very intuitive, it proves very difficult to get a close-form solution. Let's for now assume interior optimum to the IC problem, which we use the first-order approach (FOA) to tackle as follows,

$$\int_{\underline{x}}^{\overline{x}} u(w(x))f_a(x,a)dx - \psi'(a) = 0 \text{ and } \int_{\underline{x}}^{\overline{x}} u(w(x))f_{aa}(x,a)dx - \psi''(a) = 0.$$

If we use μ as the multiplier on the effort first-order condition (associated with the IC constraint), we can set up the Lagrangian to the principal's optimization as follows,

$$\begin{aligned}\mathcal{L} = \textstyle\int_{\underline{x}}^{\overline{x}} v(x-w(x))f(x,a)dx + \lambda\left[\int_{\underline{x}}^{\overline{x}} u(w(x))f(x,a)dx - \psi(a) - u\right] \\ +\mu\left[\int_{\underline{x}}^{\overline{x}} u(w(x))f_a(x,a)dx - \psi'(a)\right].\end{aligned}$$

If we do a maximization pointwise inside integral (iso-parametric problem in calculus of variation), the optimal condition is,

$$\frac{v'(x-w(x))}{u'(w(x))} = \lambda + \mu \cdot \frac{f_a(x,a)}{f(x,a)} \ \forall x,$$

which is often called the "*modified Borch rule*." Note that the second term on the right hand side measures how much risk-sharing is sacrificed for incentive reasons. (Explain in more detail on how it does the job. Possibly refer back to the paper.) Also note that

$$\frac{f_a(x,a)}{f(x,a)} = \frac{\partial \ln f(x,a)}{\partial a}$$

measures the marginal change in log-likelihood of output with respect to a, i.e., the likelihood ratio in response to one unit change in effort levels. A positive likelihood ratio would indicate that additional effort would make the current output level more likely and vice versa. A zero likelihood ratio indicates that additional effort doesn't help to change the likelihood of the current output level, i.e., the likelihood ratio is least informative about the effort content in the output.

Theorem 1: [Holmstrom (1979)] Assume that the first-order approach is valid, then at the optimum we have $\mu > 0$, i.e., the effort first-order condition associated with the IC constraint is binding.

A few important points to note here. (1) The first-order approach (FOA) is not always valid and we are going to provide the sufficient condition for the validity of FOA shortly. For instance, if the principal wants to elicit the least costly action in A, then the optimum may well be at the corner and in this case $\mu = 0$. (An example?) (2) It is a crucial assumption that the support of $f(x, a)$ doesn't change with a. In fact, if the lower-end point of support, x, shifts with a, then the first best solution may be achieved. (An example?) (3) The principal's commitment to enforce the contract is important. Again note the ex ante efficiency and ex post inefficiency of commitment.

We may be interested in the **monotonicity** of the wage schedule $w(x)$. However, the output level x is informative in conveying information about the effort level a, and we are not necessarily interested in output level per se. For example, suppose that one particular moderate output level is achieved only by the agent's low effort level, and that one particular low output level is achieved no matter how much effort the agent puts. If it is the principal's goal to encourage high effort level, then it is optimal to pay the agent more in low output states than in moderate output states, even though the principal prefers moderate output levels to low ones.

Definition: The monotone likelihood ratio property (MLRP) is satisfied for $f(x, a)$ and $F(x, a)$ if and only if the following holds,

$$\frac{d}{dx}\left(\frac{f_a(x, a)}{f(x, a)}\right) \geq 0.$$

If it is the case that $A \equiv \{a_L, a_H\}$ is non-differentiable, then MLRP requires,

$$\frac{d}{dx}\left(\frac{f(x, a_H) - f(x, a_L)}{f(x, a_H)}\right) \geq 0.$$

Intuitively, MLRP says that at higher output level, the output level is more informative about the effort level.

What's the relationship between MLRP and FOSD? It turns out that MLRP implies FOSD, but the converse is not true. Here we prove that MLRP implies FOSD. Recall that FOSD is $F_a(x, a) < 0$, $\forall x \in (\underline{x}, \overline{x})$. We can rewrite it as

$$F_a(x,a) = \int_{\underline{x}}^{x} \frac{f_a(s,a)}{f(s,a)} f(s,a)ds.$$

We know that when $x = \overline{x}$,

$$F_a(x,a) = \frac{\partial}{\partial a}\left[\int_{\underline{x}}^{\overline{x}} \frac{f_a(s,a)}{f(s,a)} f(s,a)ds\right] = \frac{\partial 1}{\partial a} = 0.$$

When $x < \overline{x}$, the integral must be negative in that MLRP says that $f_a(x,a)/f(x,a)$ is increasing in x. Hence, MLRP implies FOSD. The fact that the converse is not true implies that FOSD is a weaker concept than MLRP.

Theorem 2: [Holmstrom (1979) and Shavell (1979)] Under the first-order approach (FOA), if $f(x,a)$ satisfies MLRP, then $w(x)$ is increasing in x.

This is actually a very useful theorem when constructing models. Let's use an example to illustrate its use. Suppose that there are two possible effort levels, a_H or a_L, and three possible output level, x_1, x_2, x_3, where $x_1 < x_2 < x_3$. The probability density $f(x,a)$ is distributed as follows.

$f(x,a)$	x_1	x_2	x_3
a_H	0.4	0.1	0.5
a_L	0.5	0.4	0.1
$\frac{f(x,a_H)-f(x,a_L)}{f(x,a_H)}$	-0.25	-3.0	0.8

If the principal wishes to induce high effort in this case, he must use a non-monotonic[2] wage schedule to: (1) punish moderate outputs which are most indicative of low effort (i.e., highest likelihood ratio in absolute value, a higher effort level reduces the probability of moderate output); (2) reward high outputs which are quite informative about high effort; and (3) provide moderate income for low outputs which are not very informative (i.e., lowest likelihood ratio in absolute value). The conclusion is that the optimal wage schedule is not driven by output, but by the likelihood ratio, i.e., by the informativeness of output about effort level. One real world application is that when designing executive contract, we should filter out the systematic risk component and reward on the idiosyncratic component.

[2] Since the likelihood ratio doesn't increase monotonically with respect to output level, i.e., MLRP doesn't hold, the optimal wage schedule need not be monotonic.

In terms of validity of first-order approach (FOA), Mirrlees (1974) and (1976) show that first-order approach is not generally valid. Grossman-Hart (1983) provides the following:

Theorem 3: The combination of monotone likelihood ratio property (MLRP) and convexity of distribution function condition (CDFC) is sufficient for first-order approach to be valid and thus for $w(x)$ to be monotonic.

Definition: *Convexity of Distribution Function Condition (CDFC)* A distribution satisfies CDFC if and only if the following holds,

$$F[x, \alpha \cdot a + (1-\alpha) \cdot a'] \leq \alpha \cdot F(x,a) + (1-\alpha) \cdot F(x,a'), \forall \alpha \in [0,1].$$

The condition above is essentially saying $F_{aa} \geq 0$.

If CDFC holds, then the agent's objective function becomes globally concave in a for any wage schedule $w(\cdot)$. For example, a generalized uniform (a form of Beta distribution) will satisfy both MLRP and CDFC. $F(x,a) = \left(\frac{x-\underline{x}}{\overline{x}-\underline{x}}\right)^{1/(1-a)}$, where $a \in A \equiv [0,1)$.

4.2 A Special Principal-Agent Problem with Closed-form Solution

4.2.1 Holmstrom and Milgrom (1987)

In an effort to justify why there are so many linear contracts in the real world, the authors design the following explicit model. An agent continuously varies his effort and observes continuous output. Output is modeled as a Brownian motion with drift μ. The agent's utility is exponential, $u(w,\mu) = -\exp\{-r[w - c(\mu)]\}$, where r is the CARA parameter. The principal is assumed to be risk-neutral and $dx = \mu(a)dt + \sigma dz$, where x is an arithmetic Brownian motion with drift $\mu(a)$ and constant volatility σ. Let the profits be $x(t) = \mu + \tilde{\varepsilon}$, and the terminal distribution of $x(T) = \mu(a) + \tilde{\varepsilon}$ is simply $N(0, \sigma^2 T)$. The disutility of effort is $c(\mu) = \frac{1}{2}k\mu^2$ and the reservation utility level u is assumed to be 0.

It is easy to work out the first-best solution using the knowledge we have mentioned in the previous section. The solution is

$$\mu^{fb} = \tfrac{1}{k}; w^{fb} = \tfrac{1}{2k}; u^{fb} = \tfrac{1}{2k} - \tfrac{k}{2} \cdot \tfrac{1}{k^2} = 0,$$

and the principal split the output with the agent in a 50-50 fashion.

In terms of the second-best solution, we show that the CARA utility and normal distribution of $x(T)$, which comes from the arithmetic Brownian motion, ensure the linearity of the optimal contract, which is denoted here as $w(x) = \alpha x + \beta$.

The moment-generating function for a normal distribution is $M_x(t) = \exp(\mu t + \frac{1}{2}\sigma^2 t^2)$ and $E_x(e^{tx}) = M_x(t)$. Therefore, we have

$$E\left\{-\exp\{-r[\alpha\mu + \alpha\varepsilon + \beta - c(\mu)]\}\right\} = -\exp\{-r[\alpha\mu + \beta - c(\mu)] + \tfrac{1}{2}\sigma^2 r^2 \alpha^2\}.$$

The agent's utility function is over $\alpha x + \beta - c(\mu)$. Now when the agent's utility is $\exp(\mu t + \frac{1}{2}\sigma^2 t^2)$, the certainty equivalent amount will be $\mu + \frac{1}{2}\sigma^2 t$ and here $t = -r$.

The certainty equivalent amount is

$$\alpha\mu + \beta - c(\mu) - \tfrac{1}{2} r\sigma^2\alpha^2 = \alpha\mu + \beta - \tfrac{1}{2}k\mu^2 - \tfrac{1}{2}r\sigma^2\alpha^2.$$

Hence the agent's problem is

$$\max_{\mu} \alpha\mu + \beta - \tfrac{1}{2}k\mu^2 - \tfrac{1}{2}r\sigma^2\alpha^2.$$

The first-order condition on μ is $\alpha - k\mu = 0$ and thus $\mu = \alpha/k$. (Note that the effort level positively depends on the contract slope α. Interpret it intuitively.)

The principal maximizes $x - w(x) = x - \alpha x - \beta$. So the principal and agent jointly maximize the following objective[3],

$$\alpha\mu + \beta - \tfrac{1}{2}k\mu^2 - \tfrac{1}{2}r\sigma^2\alpha^2 + x - \alpha x - \beta = \mu + (1-\alpha)\varepsilon - \tfrac{1}{2}k\mu^2 - \tfrac{1}{2}r\sigma^2\alpha^2,$$

since $x = \mu + \varepsilon$. The first-order condition to this optimization implies $\mu = 1/k$. Note that β doesn't play a role here. The only role β plays is to ensure the IR constraint.

Taking expectation in the joint objective, we see the principal maximizing

$$\max_{\alpha,\mu} \mu - \tfrac{1}{2}k\mu^2 - \tfrac{1}{2}r\sigma^2\alpha^2 \text{ s.t. } \alpha = \mu k.$$

The solution is (α^*, μ^*, π^*), where π^* is the net profit, as follows:

$$\alpha^* = \frac{1}{1 + kr\sigma^2}; \mu^* = \frac{\alpha^*}{k} = \frac{1}{k(1 + kr\sigma^2)}.$$

[3] We do this to find out the difference between the first-best solution and the second-best solution.

Note that $\alpha^* < 1$ since $kr\sigma^2 > 0$. Furthermore, we have $\mu^* = \alpha^* \cdot \mu^{fb} < \mu^{fb}$ since $\alpha^* < 1$. (Do some comparative statics and interpret the results intuitively.) The authors actually proved that the agent will adopt a constant effort level in a very general case.

The agent's certainty equivalent, $\alpha\mu + \beta - \frac{1}{2}k\mu^2 - \frac{1}{2}r\sigma^2\alpha^2$, is equal to the reservation utility level 0. Hence we have

$$\alpha\mu + \beta = \tfrac{1}{2}k\mu^2 + \tfrac{1}{2}r\sigma^2\alpha^2 = \tfrac{1}{2k(1+kr\sigma^2)}.$$

The expected wage is then

$$w(x) = \alpha\mu + \beta = \frac{1}{2k(1+kr\sigma^2)},$$

and thus the net profit is

$$\pi^* = \mu^* - \alpha^*\mu^* - \beta^* = \frac{1}{2k(1+kr\sigma^2)}.$$

4.3 References

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Chapter 5

Dynamic Games and Reputation

5.1 Holmstrom and Ricart I Costa (1986)

In this type of models, we are dealing with managerial incentives and capital management. In particular, the relevant moral hazard for senior executives is not laziness but investment distortions (either over-investment or under-investment) that were used to affect ability perceptions. The distortion propensity may create a need for capital rationing and hence centralized capital budgeting. This type of model is also known as "Reputational Concerns Model" or "Career Concerns Model." The difference between the moral hazard here from the adverse selection in Akerlof's paper is that people opt in or out based upon the contracts at choice, and that we replace effort with project choice here. Reputation models usually have more than one period in consideration, while adverse selection models deal with one-shot game.

Suppose that there is a risk-neutral firm with a risk-averse manager. There are two periods in consideration, with consumption c_1 and c_2 in each period. The manager's utility function is $u(c_1, c_2) = u(c_1) + \beta u(c_2)$. At time $t = 0$, the manager makes a decision on investment plan 1 that pays off at time $t = 1$. At the time $t = 1$, the manager makes another decision on investment plan 2 that pays off at time $t = 2$. We put investments in two separate periods here because there is a reason to hire the manager for two periods. The investment payoffs in two periods are $y_t, t = 1, 2$. The manager receives a signal s_t that predicts the payoff y_t in the following fashion: $y_t = s_t + \varepsilon_t$. We assume that the signal s_t is independent of the manager's ability, and the noise term ε_t is correlated with manager's ability and realized after investment decisions were made.

In a discrete time setting, we allow the signal and the noise term taking a value of either 1 or –1, hence the payoff level takes the value of –2, 0 or 2. Suppose that the manager could be of either good or bad, denoted by $\tau \in \{g, b\}$. Furthermore, we assume the following priors:

$$\Pr(s_t = 1) = \Pr(s_t = -1) = \tfrac{1}{2};$$
$$\Pr(\varepsilon_t = 1|\tau = g) = \tfrac{3}{4}, \Pr(\varepsilon_t = 1|\tau = b) = \tfrac{1}{4};$$
$$\Pr(\varepsilon_t = -1|\tau = g) = \tfrac{1}{4}, \Pr(\varepsilon_t = -1|\tau = b) = \tfrac{3}{4}.$$

Assume that there is symmetric uncertainty about the manager's ability, i.e., $\Pr(\tau = g) = p_1$. After the decision over investment plan 1 is made, the payoff y_1 and signal s_1 are publicly known and thus ε_1 is inferred. People would update their beliefs about the manager's ability using the inferred noise term. The manager is paid w_1 and w_2 at the beginning of time $t = 1$ and $t = 2$, respectively. Note that no contingent contract is allowed in this model; contracts are set period by period. Therefore, w_1 doesn't depend upon y_1, but w_2 will reflect people's updated belief. "Reputation" in this model is the belief about the manager's ability.

At time $t = 0$, the manager observes signal s_1 and decides whether or not to make investment for plan 1. Since the signal s_1 doesn't reveal information about the manager's ability, the market and the manager have symmetric information in this sense. At the time $t = 1$, the manager gets the wage w_1 and the public observes y_1 and s_1. The public update the prior belief p_1 in the following way: (1) if the manager made the investment and the project turned out to be a success (measured by $\varepsilon_1 = 1$), then the updated belief is $p_2^+ = \Pr(\tau = g|s_1, \varepsilon_1 = 1, invest)$; (2) if the manger made the investment and the project turned out to be a failure (measured by $\varepsilon_t = -1$), then the updated belief is $p_2^- = \Pr(\tau = g|s_1, \varepsilon_1 = -1, invest)$; (3) if the manager didn't make the investment, then the updated belief is $p_2^0 = \Pr(\tau = g|s_1, \varepsilon_1, not\ invest)$.

The derivations are omitted here and we concentrate on the major results.

Proposition 1: If the manager is risk-averse, then he will never invest. (under-investment)

There are two key facts/assumptions supporting this result. One is the fact that beliefs form a martingale, i.e., $E(p_2) = p_1$. It says that the expected reputation (and thus wage) if the manager invests is the same as the expected reputation if s/he doesn't invest. The other is the assumption that the noise term ε_t is observed ex post so that the value of signal s_t is reputationally irrelevant in that s_t doesn't provide information about ε_t. The manager only pays attention to the noise term ε_t because the wage depends upon the reputation.

Proposition 2: [Optimal Multi-period Contract, Harris and Holmstrom, RES, 1982]

An optimal 2-period wage contract is downward rigid (because of risk-aversion).

This result says that the wage can never be lower than the wage in the period before. In the 2-period context, it means wage w_1 provides a floor for w_2 and thus effectively increases w_2 relative to w_1. Also note the so-called "quitting constraint," which says that only the firm is obligated to honor the contract and the manager can always quit. If the manager was found to be a bad manager after period 1, the firm has to pay no less in the next period. If the manger was found to be a good manager after period 1, the manager always has the option to quit to seek the market wage dependent upon the market inferred belief about his ability. Hence the wage contract at period 2 is effectively the wage rate at the previous period plus a call option.

Proposition 3: If the manager is not too risk-averse, then the option he has on his human capital will cause him to over-invest.

We say the manager is "not too risk-averse" in the sense that there is a tradeoff between the convex wage schedule and the concave utility function. If the impact from the convex wage schedule is greater than that from the concave utility function, then the manager will over-invest. Furthermore, the over-investment will generally lead to capital rationing. In this particular case, the manager can still benefit from the option even if he doesn't invest in the first period.

5.2 Milbourn, Shockley and Thakor (2001)

5.3 Diamond (1991)

In an effort of addressing how firms decide to borrow money, either by issuing commercial papers or by borrowing from banks, Diamond provides a demand theory for bank loans. The key result is that firms with average credit ratings tend to use bank loans while firms with high credit ratings borrow directly. The intuition behind this result hinges upon a reputation effect of bank loans. Firms that borrow from banks are subject to a costly monitoring process, the result of which contributes to the credit ratings firms receive and will affect firms' future cost of capital.

Borrowers can be one of three types, type G who invests in a safe project each period with sure return G that is higher than the riskless rate R, type B who invests in a risky project in each period that yields return B with probability π and zero return with probability $1 - \pi$, or type BG who has a choice of action $a_t \in \{g, b\}$ at period t. A type BG borrower who chooses action $a_t = g$ invests in the safe project and an action of $a_t = b$ leads to the payoff from the risky project. Although the return B is higher than the return G, the expected return on the risky project is lower than the riskless rate, i.e., $\pi B < R$ and $B > G$. Borrowers' true type is their private information.

Each project requires \$1 of fund and lasts for one period. At the beginning of the period t, each borrower presents a debt contract with face value r_t to the lenders, who decide which loans to monitor at the expense of C per project period. Type BG borrowers also get a chance to choose their action $a_t \in \{g, b\}$. The lenders' monitoring can detect with probability P only type BG borrowers who select risky projects, but doesn't result in a conclusive report on type G borrowers, type B borrowers, or type BG borrowers who select safe projects. Only when type BG borrowers were caught choosing risky projects were the loan requests turned down. The approved loans are issued and the chosen projects are implemented. The promised face value is paid at the end of period t unless the borrower's project doesn't yield enough to pay back the loan, in which case a default occurs and it appears on the borrower's track record. Also on the track record are the date on which the face value of debt was paid and previous monitoring results, if the borrower was ever monitored. Any borrowers who default at any date or are caught choosing risky projects have their credit cut off permanently. The debt contract is enforced by a highly inefficient bankruptcy court. Whenever a default occurs, the return of the project is assumed to be destroyed completely, including the portion that the borrower doesn't pay to lenders. This assumption effectively eliminates the incentive for borrowers to lie about the project return.

Since a type G or B borrowers will never have a conclusive monitoring report produced so as to affect the decision on their future loan requests, only type BG borrowers are concerned with their reputation. Let V_{t+1} stand for the present value of future rents for a type BG borrower that makes optimal decisions from $t+1$ to the terminal period $T < \infty$ and has a "clean" track record up to date t. That is, the type BG borrower never defaulted before and hasn't been caught taking risky projects when monitored.

For a type BG borrower, the expected net payoff from taking the risky project with action $a_t = b$ is $(1-P) \cdot \pi \cdot (B - r_t + V_{t+1})$ (when the type BG borrower wasn't caught in monitoring and the risky project happens to deliver B), and the expected net payoff from taking the safe project with action $a_t = g$ is $G - r_t + V_{t+1}$. Hence a type BG borrower with a clean-so-far record will select the safe project if and only if

$$G - r_t + V_{t+1} \geq (1-P) \cdot \pi \cdot (B - r_t + V_{t+1}),$$

or

$$r_t \leq \frac{G - \pi(1-P)B}{1 - \pi(1-P)} + V_{t+1}.$$

If lenders don't monitor at all, i.e., $P = 0$, then safe projects are the optimal choice for the type BG borrower if and only if the face value of the debt contract is low enough, i.e., $r_t \leq (G - \pi B)/(1-\pi) + V_{t+1}$.

When do lenders monitor? Only if their expected payoff from monitoring exceeds the composite costs $R + C$. Monitoring has impact on only type BG borrowers, and serves one of two functions, either incentives or screening. When monitoring induces type BG borrowers to choose safe projects, it acts as incentives and avoids the loss $(1-\pi)r_t$ in the event of default associated with the risky projects. Let $f_{G,t}$, $f_{B,t}$, and $f_{BG,t}$ represent the population fraction of borrowers of each type at date t. The expected payoff to the lenders from monitoring as incentives is $f_{BG,t}(1-\pi)r_t$. When monitoring helps identify type BG borrowers who select the risky projects so that lenders decline these loan requests, it acts as screening and increases the expected payoff from πr_t to R. Hence the expected payoff to the lenders from monitoring as screening is $f_{BG,t} \cdot P \cdot (R - \pi r_t)$. The value of monitoring to the lender is clearly greater when it provides incentive rather than screening, since $0 < P \leq 1$ and $r_t > R$.

In presence of monitoring, reputation strengthens monitoring in the sense that type BG borrowers with lower credit ratings (higher value of r_t) are now induced to choose safe projects although they wouldn't do so had it not been reputation concerns, e.g., $\frac{G-\pi(1-P)B}{1-\pi(1-P)} < r_t < \frac{G-\pi(1-P)B}{1-\pi(1-P)} + V_{t+1}$.

The endogenously determined face value of debt corresponding to the monitoring choices can be determined as follows. For the case of no monitoring and $a_t = g$, it is obvious

$$r_t^g = \frac{R}{f_{G,t} + \pi f_{B,t} + f_{BG,t}}.$$

For the case of monitoring as incentives for inducing $a_t = g$, we have

$$r_t^I = \frac{R + C}{f_{G,t} + \pi f_{B,t} + f_{BG,t}}.$$

For the case of no monitoring and $a_t = b$, the face value is

$$r_t^b = \frac{R}{f_{G,t} + \pi(f_{B,t} + f_{BG,t})}.$$

For the case of monitoring as screens with $a_t = b$, we have

$$r_t^S = \frac{C + R(1 - Pf_{BG,t})}{f_{G,t} + \pi[f_{B,t} + (1-P)f_{BG,t}]}.$$

Because borrowers offer the lowest possible face value at each date (this is a sequential equilibrium supported by the lenders' belief that borrower type is not a function of the face value offered, see the explanation for Lemma 1), every borrower will choose the smallest of all relevant face values in the period concerned.

We can interpret $1 - f_{B,t}$ as the borrower's credit rating and $f_{BG,t}$ as the pervasiveness of moral hazard among all borrowers. After quite some algebra, Diamond shows the following results. A borrower with high enough credit rating chooses to issue commercial paper directly as monitoring is unnecessary due to lack of moral hazard. For borrowers with intermediate credit rating, monitoring provides incentives and works the best for the lenders. If moral hazard is rather pervasive, these borrowers would borrow from banks. For borrowers with still lower credit rating, monitoring acts as a screen and works less well for the lenders. If moral hazard is not pervasive enough, then monitoring costs will not be worth the cost and borrowers will issue commercial papers directly. For the case of low monitoring cost accompanied by very pervasive moral hazard, borrowers with low credit ratings use bank loans subject to screening. For the case of high monitoring cost, these borrowers won't be able to obtain fund from anywhere.

5.4 References

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Chapter 6

Models of Herding Behavior

6.1 Bikhchandanni, Hirshleifer and Welch (1992)

6.2 Prendergast (1993)

This paper studies a tradeoff between inducing workers to exert effort and leading workers to conform to the supervisor's opinion when subjective performance evaluation is used. Two major results of the paper are: the subordinate's desire for conformity leads to inefficiency and the supervisor may have an optimal balance between the two tradeoffs above.

In a pervasively risk-neutral environment, a manager (m) and a worker (w) set out to observe a parameter η, which is normally distributed with mean η_0 and variance σ_0^2. Each of the two people observes only a garbled signal about the value of the parameter, i.e., $\eta_m = \eta + \varepsilon_m$ and $\eta_w = \eta + \varepsilon_w$, where $\varepsilon_m \sim N(0, \sigma_m^2)$ and $\varepsilon_w \sim N(0, \sigma_w^2)$ are uncorrelated. Agent $i \in \{m, w\}$ exerts mutually unobservable effort e_i at the expense of $C_i(e_i)$ to uncover its respective noisy signal with variance $\sigma_i^2 = h_i(e_i)$. Assume that both the manager and the worker get more precise observations with more efforts, i.e., $h_i'(e_i) < 0$ and $h_i''(e_i) > 0$. Also assume the usual convex cost function, $C_i'(e_i) > 0$, $C_i''(e_i) > 0$. The boundary conditions for the cost functions are $C_i(0) = 0$, $C_i'(0) = 0$ and $C_i'(\infty) = \infty$.

Without involving any effort or cost, the worker observes one additional signal reflecting what the manager has seen, i.e., $\eta_\lambda = \eta_m + \lambda$, where $\lambda \sim N(0, \sigma_\lambda^2)$ is uncorrelated with all previously defined random variables.

Clearly, if η_m is known to the worker, η_λ would be ignored due to the unnecessary noise introduced by λ. It will be shown shortly that the worker doesn't ignore η_λ in the equilibrium as long as $e_w > 0$.

Let $\widehat{\eta}_w$ represent the value of η reported by the worker, who is compensated based on η_0, η_m and $\widehat{\eta}_w$, the three pieces of information the manager uses to determine the posterior belief of the true parameter η. The business around the parameter η is motivated by the goal of minimizing the posterior variance $Var(\eta|\eta_0, \eta_m, \widehat{\eta}_w)$. All variances, σ_0^2, σ_m^2, σ_w^2 and σ_λ^2, are assumed to be common knowledge.

At the first best, the worker truthfully reports η_w so that the manager's posterior belief about η is normally distributed with

$$\begin{aligned} E(\eta|\eta_0, \eta_m, \eta_w) &= \frac{\frac{1}{\sigma_0^2}\eta_0 + \frac{1}{\sigma_m^2}\eta_m + \frac{1}{\sigma_w^2}\eta_w}{\frac{1}{\sigma_0^2} + \frac{1}{\sigma_m^2} + \frac{1}{\sigma_w^2}}; \\ Var(\eta|\eta_0, \eta_m, \eta_w) &= \frac{1}{\frac{1}{\sigma_0^2} + \frac{1}{\sigma_m^2} + \frac{1}{\sigma_w^2}}. \end{aligned}$$

That is, the posterior mean is just the precision-weighted average of the prior and the manager and the worker's observations, while ignoring the garbled second-guess by the worker.

Because of the universal risk-neutrality, the worker is always compensated at the reservation utility r plus the cost of efforts exerted $C_w(e_w)$. Let the profit of the firm be negative variance of the manager's posterior belief about η. Then the manager sets the first-best efforts level by solving

$$\max_{e_m, e_w} \{-Var(\eta|\eta_0, \eta_m, \eta_w) - C_m(e_m) - [C_w(e_w) + r]\}.$$

There exists a unique solution to the optimization above since $h_i(e_i)$ is concave and $C_i(e_i)$ is convex for $i \in \{m, w\}$.

If the wage contract for the worker doesn't depend upon the worker's report, then it is natural for the worker to shirk and thus the first-best effort level is not attainable. Due to the non-observability of true η, the manager has to use η_0, η_m and $\widehat{\eta}_w$ to determine whether or not the worker has exerted any effort. Let the manager use the following wage contract: pay w_1 if $|\widehat{\eta}_w - \eta_m| < k$ and pay w_0 otherwise, where $w_1 > w_0$.

Because of this particular wage contract, the worker is motivated to exert effort and produce a report as close to the manager's observation as possible. The worker has three pieces of information, η_0, η_w and η_λ, to form an educated guess about the manager's observation η_m. First of all, the worker update his belief about η as follows,

$$E(\eta|\eta_0,\eta_w) = \frac{\frac{1}{\sigma_0^2}\eta_0 + \frac{1}{\sigma_w^2}\eta_w}{\frac{1}{\sigma_0^2}+\frac{1}{\sigma_w^2}}; Var(\eta|\eta_0,\eta_w) = \frac{1}{\frac{1}{\sigma_0^2}+\frac{1}{\sigma_w^2}}.$$

Next, the worker uses $\eta_m = \eta + \varepsilon_m$ to form a belief on η_m based upon η_0 and η_w,

$$E(\eta_m|\eta_0,\eta_w) = E(\eta|\eta_0,\eta_w); Var(\eta_m|\eta_0,\eta_w) = Var(\eta|\eta_0,\eta_w) + \sigma_m^2.$$

By incorporating the information η_λ, the worker forms a final belief

$$\begin{aligned} E(\eta_m|\eta_0,\eta_w,\eta_\lambda) &= \tfrac{E(\eta_m|\eta_0,\eta_w)/Var(\eta_m|\eta_0,\eta_w)+\eta_\lambda/\sigma_\lambda^2}{1/Var(\eta_m|\eta_0,\eta_w)+1/\sigma_\lambda^2}; \\ Var(\eta_m|\eta_0,\eta_w,\eta_\lambda) &= \tfrac{1}{1/Var(\eta_m|\eta_0,\eta_w)+1/\sigma_\lambda^2}, \end{aligned}$$

and reports to the manager

$$\widehat{\eta}_w = E(\eta_m|\eta_0,\eta_w,\eta_\lambda) = \mu_0\eta_w + \mu_1\eta_0 + \mu_2\eta_\lambda,$$

where

$$\begin{aligned} &\mu_0 \equiv \sigma_0^2\sigma_\lambda^2/\overline{\sigma}^2; \mu_1 \equiv \sigma_w^2\sigma_\lambda^2/\overline{\sigma}^2; \mu_2 \equiv (\sigma_0^2\sigma_w^2 + \sigma_0^2\sigma_m^2 + \sigma_w^2\sigma_m^2)/\overline{\sigma}^2; \\ &\overline{\sigma}^2 \equiv \sigma_0^2\sigma_w^2 + \sigma_0^2\sigma_m^2 + \sigma_w^2\sigma_m^2 + \sigma_\lambda^2(\sigma_0^2+\sigma_w^2). \end{aligned}$$

That is, at any positive effort level, the worker reports $\widehat{\eta}_w$, a distorted report that puts too little weight on his true observation η_w. The desire to conformity doesn't cause an efficiency problem when $\sigma_\lambda^2 = \infty$ since the worker's report $\widehat{\eta}_w = \mu_0\eta_w + \mu_1\eta_0$ can be used by the manager to back out the true observation η_w. It does cause efficiency loss when $\sigma_\lambda^2 < \infty$ since the manager cannot back out η_w from the report $\widehat{\eta}_w = \mu_0\eta_w + \mu_1\eta_0 + \mu_2\eta_\lambda$, due to the unobservable η_w and η_λ, the latter of which should have been useless to the manager. The best the manager can do is to invert η_w from a transformed report $z = \eta_w + \frac{\mu_2}{\mu_0}\eta_\lambda$, i.e.,

$$Var(\eta_w|\widehat{\eta}_w) = \sigma_w^2 + \left(\frac{\mu_2}{\mu_0}\right)^2\sigma_\lambda^2.$$

Therefore, the manager gets the following posterior variance

$$\begin{aligned} Var(\eta|\eta_0,\eta_m,\widehat{\eta}_w) &= \frac{1}{\frac{1}{\sigma_0^2}+\frac{1}{\sigma_m^2}+\frac{1}{\sigma_w^2+\sigma_\lambda^2(\mu_2/\mu_0)^2}} \\ &> \frac{1}{\frac{1}{\sigma_0^2}+\frac{1}{\sigma_m^2}+\frac{1}{\sigma_w^2}} \\ &= Var(\eta|\eta_0,\eta_m,\eta_w). \end{aligned}$$

The basic intuition behind is that under the subjective performance evaluation, the incentive for the worker to exert effort leads to the worker's desire for conformity, i.e., the worker partially diverts the attention from his true observation to second-guessing the manager's observation. The desire for conformity makes the worker's report less informative (with $\sigma_\lambda^2 \left(\frac{\mu_2}{\mu_0}\right)^2$ unit of additional variance) and thus a higher variance for the manager's posterior belief, comparing to the first-best level.

In short, a contract with incentives leads to dishonesty while the worker exerts positive effort, and a contract without incentives induces truth-telling while the worker exerts no effort. If σ_λ^2 is small enough, i.e., the worker can very well second guess the manager's opinion, then the manager may be better off by using contract without incentives to induce truthful reporting at a cost of zero effort.

6.3 References

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Chapter 7

Market Microstructure

7.1 Glosten and Milgrom (1985)

Here we are going to discuss a sequential trade model in a quote-driven market. The aim of the paper is to explain the existence of bid-ask spread for reasons other than inventory costs.

There is a market maker in a quote-driven system, who receives buy and sell orders. The market maker doesn't have information about the true underlying value of assets and doesn't know who has the private/true information. Moreover, the buy and sell orders reveal noisy information about the underlying value of assets. There are two types of traders in the market, one is informed traders (or inside traders) and the other is noise traders (or liquidity traders). The market maker loses money to informed traders on average and breaks even with noise traders on average. A bid-ask spread is set to allow the market maker to recoup his expected losses from trading with informed traders.

Here are some specific assumptions.
(A1) Everyone in the model is risk-neutral and the market maker is competitive.
(A2) The asset's eventual value is v, which is a random variable.
(A3) During each transaction, a trader can trade only one unit of assets and no block trades are allowed. (Note that Easley-O'Hare allow block trades to happen in their JFE paper.) Without this constraint, the informed trader will trade a large amount when time is right, which will in turn make the market maker know who has inside information.

(A4) All trades occur at either the bid or the ask price; no prices inside the spread are allowed. (They are the so-called "regret-free" prices.)
(A5) There are no transaction costs or inventory costs in the market.
(A6) Trades take place sequentially with only one trade allowed to transact at any point in time. In each round, a trader is selected probabilistically.

Now let's discuss a simplified version of the paper. Suppose that the asset's eventual value takes one of two possible values, $v \in \{\underline{v}, \overline{v}\}$. Let s_1 and b_1 represent the sell and buy events at date 1, respectively. Let α_1 and β_1 be the ask and bid prices posted. When the market maker sees a buy order, he posts the ask price in the following way,

$$\alpha_1 = E(v|b_1) = \underline{v} \cdot \Pr(v = \underline{v}|b_1) + \overline{v} \cdot \Pr(v = \overline{v}|b_1).$$

Similarly, when the market maker sees a sell order, he posts the bid price in the following way,

$$\beta_1 = E(v|s_1) = \underline{v} \cdot \Pr(v = \underline{v}|s_1) + \overline{v} \cdot \Pr(v = \overline{v}|s_1).$$

All conditional beliefs are obtained by updating the priors by Bayes' rule. For example,

$$\Pr(v = \underline{v}|s_1) = \frac{\Pr(s_1|v = \underline{v}) \cdot \Pr(v = \underline{v})}{\Pr(s_1|v = \underline{v}) \cdot \Pr(v = \underline{v}) + \Pr(s_1|v = \overline{v}) \cdot \Pr(v = \overline{v})}.$$

From the beliefs updating tree above, we know that $\Pr(s_1|v = \underline{v}) = \mu + (1-\mu)\gamma^S$, $\Pr(s_1|v = \overline{v}) = (1-\mu)\gamma^S$, and thus $\Pr(s_1) = (1-\theta)\mu + (1-\mu)\gamma^S$. Similarly, we can obtain $\Pr(b_1) = \theta\mu + (1-\mu)\gamma^B$. Note that we allow the case where no trade occurs so that it is okay to have $\Pr(s_1) + \Pr(b_1) \neq 1$.

Let's now use the following calibration: $\overline{v} = 1, \underline{v} = 0, \theta = \frac{1}{2}, \gamma^B = \gamma^S = \frac{1}{2}, \mu = \frac{1}{2}$. Clearly, we have the following results:

$$\begin{gathered}\Pr(s|v = 0) = \mu + (1-\mu)\gamma^S = \tfrac{3}{4}; \Pr(v = 0) = \tfrac{1}{2};\\ \Pr(s) = (1-\theta)\mu + (1-\mu)\gamma^S = \tfrac{1}{2};\\ \Pr(v = 0|s) = \tfrac{\Pr(s|v=0)\cdot\Pr(v=0)}{\Pr(s)} = \tfrac{3}{4}.\end{gathered}$$

Similarly, we have $\Pr(v = 0|b) = \frac{1}{4}$.

Therefore, the ask price is determined as

$$\alpha = E(v|b) = 0 \cdot \Pr(v = 0|b) + 1 \cdot \Pr(v = 1|b) = 0 + 1 \cdot (1 - \tfrac{1}{4}) = \tfrac{3}{4},$$

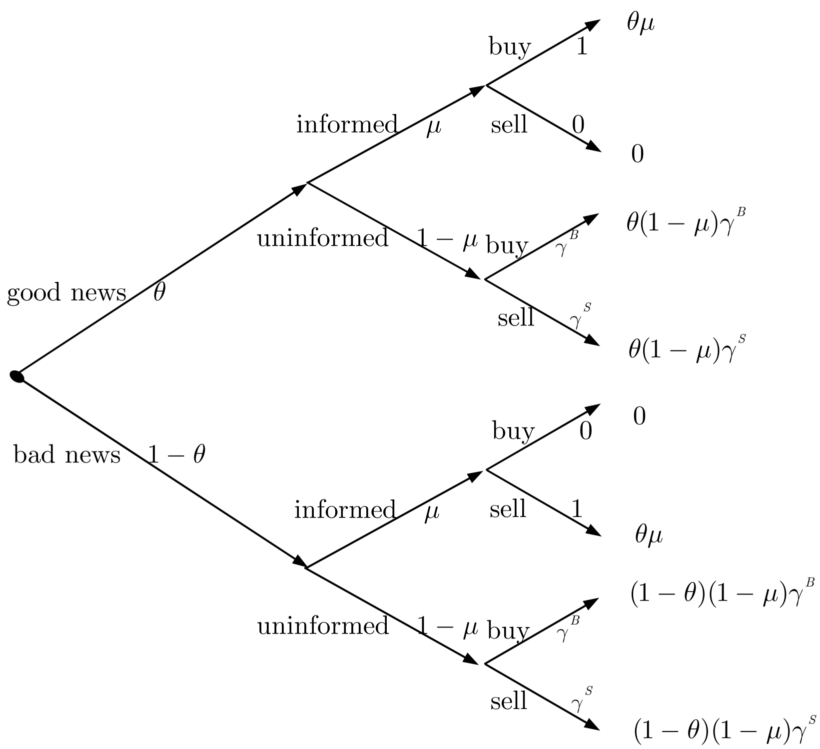

Figure 7.1: Tree of Beliefs in Glosten-Milgrom Model

and the bid price is determined as

$$\beta = E(v|s) = 0 \cdot \Pr(v = 0|s) + 1 \cdot \Pr(v = 1|s) = 0 + 1 \cdot (1 - \tfrac{3}{4}) = \tfrac{1}{4}.$$

The bid-ask spread is thus $\frac{1}{2}$.

Next, we are to find out the dynamics of price and spread. Suppose that the first trade is a buy with price $E(v|b)$, and the market maker's new prior is

$$1 - \Pr(v = 0|b) = \Pr(v = 1|b) = \tfrac{3}{4}.$$

Case (1): Suppose the second trade is a buy. The beliefs are updated as follows,

$$\Pr(v = 0|b_1, b_2) = \tfrac{\Pr(b_2|v=0,b_1)\cdot\Pr(v=0|b_1)}{\Pr(b_2|v=0,b_1)\cdot\Pr(v=0|b_1)+\Pr(b_2|v=1,b_1)\cdot\Pr(v=1|b_1)}.$$

Since

$$\begin{gathered}\Pr(b_2|v = 0) = \Pr(b|v = 0) = \tfrac{1}{4};\\ \Pr(v = 0|b_1) = \Pr(v = 0|b) = \tfrac{1}{4};\\ \Pr(b_2|v = 1) = \Pr(b|v = 1) = \mu + (1 - \mu)\gamma^B = \tfrac{3}{4};\\ \Pr(v = 1|b_1) = \Pr(v = 1|b_1) = 1 - \Pr(v = 0|b_1) = \tfrac{3}{4},\end{gathered}$$

we have

$$\Pr(v = 0|b_2, b_1) = \frac{\frac{1}{4} \times \frac{1}{4}}{\frac{1}{4} \times \frac{1}{4} + \frac{3}{4} \times \frac{3}{4}} = \tfrac{1}{10}.$$

It says that upon seeing two buy orders, it is highly unlikely that the asset is a low value asset; i.e., $\Pr(v = 1|b_2, b_1) = \frac{9}{10}$.

Case (2): Suppose the second order is a sell. The beliefs are updated as follows,

$$\Pr(v = 0|b_1, s_2) = \tfrac{\Pr(s_2|v=0,b_1)\cdot\Pr(v=0|b_1)}{\Pr(s_2|v=0,b_1)\cdot\Pr(v=0|b_1)+\Pr(s_2|v=1,b_1)\cdot\Pr(v=1|b_1)}.$$

Clearly, we have

$$\begin{gathered}\Pr(s_2|v = 0) = \Pr(s|v = 0) = \tfrac{3}{4}; \Pr(v = 0|b_1) = \Pr(v = 0|b) = \tfrac{1}{4};\\ \Pr(s_2|v = 1) = \Pr(s|v = 1) = \tfrac{1}{4}; \Pr(v = 1|s) = \tfrac{3}{4}.\end{gathered}$$

Therefore we have $\Pr(v = 0|b_1, s_2) = \frac{1}{2}$ and we are back to the prior before any trade happens. The new ask price is determined as

$$E(v|b_1, b_2) = 1 \cdot \Pr(v = 1|b_1, b_2) + 0 \cdot \Pr(v = 0|b_1, b_2) = \tfrac{9}{10},$$

and the new bid price is determined as

$$E(v|b_1, s_2) = 1 \cdot \Pr(v = 1|b_1, s_2) + 0 \cdot \Pr(v = 0|b_1, s_2) = \tfrac{1}{2}.$$

In the limiting case, a series of buy orders will lead the price to 1 and a series of sell order will lead the price to 0.

Some of the key results are summarized as follows: (1) Even without inventory cost, the paper can demonstrate the existence of bid-ask spread. (2) $E(P_{t+1}|I_t) = P_t$, where I_t is the market maker's information set. This property says that the transaction prices form a martingale. A few implications of this property are: market is semi-strong-form efficient, and first differences of the transaction price process are serially uncorrelated. (3) Sufficiently high adverse selection (for example, the probability of having an informed trader is too high.) may lead to a market breakdown.

7.2 Grossman and Stiglitz (1980)

This model is also known as "information-driven" model. There are two assets in the market, one risk-free asset with price normalized to 1, and one risky asset x with price of p. The terminal value of the risk asset is denoted as $\tilde{v}$ and it follows a normal distribution $N(\mu, \frac{1}{\rho_v})$, where ρ_v is the precision of $\tilde{v}$. There are two traders in the market, one is uninformed and the other one is informed with a signal s about $\tilde{v}$ where $s \sim N(v, \frac{1}{\rho_s})$, where v is a realization of $\tilde{v}$ and ρ_s is the precision of the signal s. Each trader is endowed with m units of risk-free asset and x^i units of risky asset. In particular, $x^i \sim N(0, \frac{1}{\rho_x}), \forall i = 1, 2$. The total random endowment of risky asset is $x = x^1 + x^2$. Then agent's utility function is $u(w) = -\exp(-w^i)$, where w^i is the i^{th} trader's wealth level. The linear conjecture of the price level for the risky asset is $p = \alpha\mu + \beta s - \gamma x$, where α, β and γ are positive constants in that higher asset mean value μ, higher private signal s, or lower endowment of risky asset will lead to higher price p.

Here we introduce some tool on working with normal distributions. Suppose that we have $f(x|\mu) = N(\mu, \sigma_x^2)$ and $g(\mu) = N(m, \sigma_\mu^2)$. Suppose further that μ is unknown, and σ_μ^2 and σ_x^2 are known. The posterior assessment of μ becomes

$$g(\mu|x) = \frac{f(x|\mu)g(\mu)}{\int f(x|\mu)g(\mu)d\mu}.$$

Specifically, the posterior is a normal distribution

$$N\left[\frac{\frac{m}{\sigma_\mu^2}+\frac{x}{\sigma_x^2}}{\frac{1}{\sigma_\mu^2}+\frac{1}{\sigma_x^2}}, \frac{1}{\frac{1}{\sigma_\mu^2}+\frac{1}{\sigma_x^2}}\right].$$

Therefore, the new mean (posterior) is a weighted average of x and m where the weights are the relative precision, and the new variance (posterior) is the inverse of the new precision which is the sum of precisions.

In our study here, the prior on v is $N(\mu, \frac{1}{\rho_v})$ and the posterior on v after observing s is normally distributed with mean $\frac{\mu\rho_v+s\rho_s}{\rho_v+\rho_s}$ and variance $\frac{1}{\rho_v+\rho_s}$.

Here comes a seemingly strange assumption in the model. Both the informed and uninformed traders see the price before submitting their order, and they will repeat this process until the price converge to the equilibrium price level. Essentially, traders based their submission on the equilibrium price.

From the linear conjecture $p = \alpha\mu + \beta s - \gamma x$, we know that the best guess about s for the uninformed is $(p-\alpha\mu)/\beta$ since $E(x) = 0$. Let's define $\theta \equiv (p-\alpha\mu)/\beta$ so that $\theta = s - (\gamma/\beta)x$ holds. Clearly we have $\theta|v \sim N\left(v, \frac{1}{\rho_s} + \left(\frac{\gamma}{\beta}\right)^2 \cdot \frac{1}{\rho_x}\right)$. Let's define $\frac{1}{\rho_\theta} \equiv \frac{1}{\rho_s} + \left(\frac{\gamma}{\beta}\right)^2 \cdot \frac{1}{\rho_x}$. The uninformed's posterior belief about the asset value is normally distributed, i.e., $v|\theta \sim N(\frac{\mu\rho_v+\theta\rho_\theta}{\rho_v+\rho_\theta}, \frac{1}{\rho_v+\rho_\theta})$.

The optimization problem for the uninformed is then

$$\max_y -\exp(-w^i) \text{ s.t. } w^i = vy + (m-py).$$

In the budget constraint above, vy is the investment in the risky asset and $m - py$ is the endowment of risk-free assets. This optimization can be rewritten as

$$\max_y \int -\exp[-vy-(m-py)]q(v|\theta)dv, \text{ where } p = \alpha\mu + \beta s - \gamma x.$$

Using the fact that $E[\exp(n)] = \exp[E(n) + \frac{1}{2}(n)]$ for a normally distributed variable n, the objective can be rewriteen as

$$\max_y -\exp[-y \cdot E(v|\theta) - (m-py) + \tfrac{1}{2}y^2 \cdot (v|\theta)].$$

The informed has exactly the same optimization problem except that θ was replaced by s. The first-order condition yields the informed's and uninformed's demand as

$$D^I = \frac{E(v|s) - p}{Var(v|s)}; D^U = \frac{E(v|\theta) - p}{Var(v|\theta)}.$$

If we substitute previous results into these demand equations, we have

$$D^I = \mu\rho_v + s\rho_s - p(\rho_v + \rho_s); D^U = \mu\rho_v + s\rho_\theta - p(\rho_v + \rho_\theta).$$

By equating supply to demand, $D^I + D^U = x$, we have

$$p = \tfrac{2\mu\rho_v + s\rho_s + \theta\rho_\theta - x}{2\rho_v + \rho_s + \rho_\theta}.$$

If we substitute $\theta = s - \left(\frac{\gamma}{\beta}\right) x$ into this solution, we have

$$p = \tfrac{2\mu\rho_v + s(\rho_s + \rho_\theta) - x\left[1 + \left(\frac{\gamma}{\beta}\right)\rho_\theta\right]}{2\rho_v + \rho_s + \rho_\theta}.$$

We can solve the following system of equations for $\alpha, \beta, \gamma, \rho_\theta$,

$$\begin{aligned}
\alpha &= \tfrac{2\rho_v}{2\rho_v + \rho_s + \rho_\theta}; \\
\beta &= \tfrac{\rho_s + \rho_\theta}{2\rho_v + \rho_s + \rho_\theta}; \\
\gamma &= \tfrac{1 + \left(\frac{\gamma}{\beta}\right)\rho_\theta}{2\rho_v + \rho_s + \rho_\theta}; \\
\tfrac{1}{\rho_\theta} &= \tfrac{1}{\rho_s} + \left(\tfrac{\gamma}{\beta}\right)^2 \cdot \tfrac{1}{\rho_x}.
\end{aligned}$$

The equilibrium price is certainly linear, as conjectured before.

There are a number of weakness of the paper: (1) the price conjecture is linear; (2) conditioning trades on equilibrium price; (3) non-strategic behavior of the informed; (4) possibility of negative price.

7.3 Kyle (1985)

Here are a few major distinctions between Kyle (1985) and Grossman and Stiglitz: (1) the model in this paper doesn't assume "conditioning trades on price;" (2) it assumes risk-neutrality and still gets interior solution. This model is also known as "strategic-trader" model. Both the Grossman and

Stiglitz and the Kyle models are batch-clearing model in an order-driven market.

In the market of one asset, there are two traders: the informed trader submits trade quantity x and the noise trader submits trade quantity $y \sim N(0, \sigma_y^2)$. The informed trader receives a private signal v about the underlying value of the asset, where $v \sim N(p_0, \Sigma_0)$. The informed trader knows v and the distribution of y and submits order x, while the market maker observes the order flow $\theta \equiv x + y$. Every participant, including the market maker, is risk-neutral in this model. The market marker earns zero profit while the informed trader earns profit $\pi = (v - p)x$, where p is the price set by the market maker.

Suppose that the market maker conjectures that the informed trader will follow a linear order strategy given by $x = \beta(v - p_0)$. (We are going to verify later on that this linear strategy exists.) Then we have $\theta = \beta(v - p_0) + y$, i.e., $\frac{\theta}{\beta} + p_0 = v + \frac{y}{\beta}$. We can define a transformation of the order flow as $z \equiv \frac{\theta}{\beta} + p_0$. It is clear that $z = v + \frac{y}{\beta}$ and thus $z|v \sim N(v, \frac{\sigma_y^2}{\beta^2})$. Given the observation of z, the market marker's posterior belief about v is normally distributed, i.e., $v|z \sim N(p_1, \Sigma_1)$, where

$$p_1 = \frac{\frac{p_0}{\Sigma_0} + \frac{z}{\sigma_y^2/\beta^2}}{\frac{1}{\Sigma_0} + \frac{1}{\sigma_y^2/\beta^2}}, \text{ and } \Sigma_1 = \frac{1}{\frac{1}{\Sigma_0} + \frac{1}{\sigma_y^2/\beta^2}}.$$

It is clear that the market maker is effectively adopting a linear pricing rule so that the informed trader can conjecture it as $p_1 = p_0 + \lambda(x + y)$, i.e., the informed trader takes into account the impact of his trade on the equilibrium price. Hence the informed trader can solve his own optimization problem as

$$\max_x E(\pi) = \max_x E[(v - p)x], \text{ where } p = p_1.$$

The first order condition of this problem is $v - p_0 - 2\lambda x - \lambda E(y) = 0$, i.e.,$x = \frac{v - p_0}{2\lambda}$. Since in the original conjecture, we have $x = \beta(v - p_0)$, the comparison of the conjecture and the solution of x yields $\beta = \frac{1}{2\lambda}$. Similarly, we can compare the conjecture and the solution of p_1 to get $\lambda = \frac{1}{2}\left(\frac{\Sigma_0}{\sigma_y^2}\right)^{\frac{1}{2}}$ and $\beta = \left(\frac{\sigma_y^2}{\Sigma_0}\right)^{\frac{1}{2}}$.

One simple implication is that the informed trades more when the noise trading has more variance. On the other hand, if there is no randomness at all for the noise trader, then the informed will not trade at all.

7.4 References

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Chapter 8

Security Design

8.1 Boot and Thakor (1993)

The authors are intrigued by two questions in the security design literature. First, why would a firm partition its cash flow across multiple types of financial claims? Second, why do firms pool individual assets into a portfolio and then partition the portfolio cash flows?

The authors' intuition goes as follows. A revenue-maximizing firm is willing to split its cash flow into two parts, one senior and information-insensitive component as well as one subordinate yet information-sensitive component. That's because wealth-constrained informed investors would be able to concentrate in only the information-sensitive part (they earn zero expected profit from trading the other part), rather than being forced to invest in the composite asset. Such a split effectively boosts up the informed leverage in the information-sensitive component of the firm and the higher informed demand helps drive the asset price closer to its true value. Obviously, only the high-valued firm would benefit from so doing, but the low-valued firm couldn't afford not doing so.

As far as the practice of bundling and then re-partitioning individual cash flows is concerned, it benefits a revenue-maximizing high-valued firm because the formation of a portfolio diversifies away the noisiness of signals on the individual asset values. Less noisy signals to informed investors entice higher informed demand for the asset, and thus once again the individual asset's true value (that is, the asset value being high) becomes less opaque.

We consider first the case of one firm offering one unit of asset, the value of which could be high, $\tilde{x} = \overline{x}$, or low, $\tilde{x} = \underline{x}$. The true type of the firm (either G for the high-valued or B for the low-valued) is not in the domain of public information. The common prior is that any firm is of type G with probability q. Three types of risk-neutral investors are involved in the market. Purely liquidity traders demand l dollar worth of asset, the randomness of which is captured by the probability density function $f(l)$ over $(0, \infty)$. Each atomistic informed trader spends M dollar to acquire a perfect signal ϕ about the firm's true type, and has one dollar remaining in wealth to invest on the asset if the firm turns out to be type G, i.e., $\phi = G$. The signal is identical, or "photocopied," across all informed traders. There also exist sufficient number of uninformed discretionary traders (UDTs) jointly acting as a market maker in taking the residual demand and setting the clearing price so that the expected profit for one UDT to become informed at the expense of M dollar is zero. This zero expected profit condition effectively determines the actual number of informed traders in the market who submit θ dollar worth of order in total in the event of observing a "photocopied" signal that reveals the firm as type G. The price impact of each informed trader is assumed to be negligible.

Let V represent a marginal investor's expected payoff to become informed. We can write

$$V = -M + q \int_0^\infty \frac{\overline{x} - P^e(\theta + l)}{P^e(\theta + l)} f(l) dl,$$

where $P^e(D(\phi, l))$ is the equilibrium price of the asset when the demand (in dollar term) from purely liquidity traders and informed traders is $D(\phi, l) = \theta + l$ upon the photocopied signal $\phi = G$ revealed to informed traders. Note that no short sale is allowed in the current setup and the investor is restrained in wealth (with one dollar remaining to invest on the asset upon receiving a good signal); otherwise, the informed traders would hold infinite position. The endogenously determined measure of informed traders θ^* can be solved from the zero-expected-profit condition for the marginal informed investor,

$$V(\theta^* | q, x, \overline{x}, M, f(l)) = 0.$$

The equilibrium price is determined by

$$\begin{aligned} P^e(D(\phi, l)) &= \overline{x} \cdot \Pr(\phi = G | D(\phi, l)) + \underline{x} \cdot \Pr(\phi = B | D(\phi, l)) \\ &= [\overline{x} - \underline{x}] \cdot \Pr(\phi = G | D(\phi, l)) + \underline{x}, \end{aligned}$$

where the conditional probability $\Pr(\cdot|\cdot)$ is calculated using Bayes' rule

$$\Pr(\phi = G | D(\phi, l)) = \frac{q \cdot f(D - \theta)}{q \cdot f(D - \theta) + (1 - q) \cdot f(D)}.$$

The revenue for the type G firm for providing one unit of asset can be written as

$$R^G \equiv \int_0^\infty P^e(\theta + l) f(l) dl = \underline{x} + \int_0^\infty [\overline{x} - \underline{x}] \cdot \Pr(\phi = G|\theta + l) f(l) dl.$$

The authors impose a crucial assumption $f'(\theta{+}l) < 0, \forall l \in (0, \infty)$, which is sufficient to reach two intuitive results, $\partial P^e/\partial\theta > 0$ and $\partial V/\partial\theta < 0$. That is, the higher is the informed demand, the higher the equilibrium price is. On the other hand, the higher informed demand (i.e., more competition among informed), the lower expected profit for the marginal informed investor. It is also easy to see that $\partial R^G/\theta > 0$ under this assumption, listed as Proposition 1.

Proposition 1: The type G issuing firm's equilibrium expected revenue is increasing in the measure of the set of informed traders, θ. The type B firm's expected revenue is decreasing in θ.

Since the expected revenue of type G issuing firm is increasing in the informed demand, this type of issuing firm should design the security so as to induce more traders to become informed. This goal can be achieved by decomposing securities. In particular, the issuer can split the composite security into two securities: a senior security A that is not information sensitive and guarantees a payoff of x, and a junior security S that is more information sensitive than the composite security and pays off $\overline{x} - \underline{x}$ by a type G firm and 0 by a type B firm.

Gorton and Pennacchi (1990) show that once a less-information-sensitive security is created the liquidity demand will be shifted toward that security. Boot and Thakor assume that there is still sufficient liquidity demand left for the information sensitive security so as to guarantee positive informed demand for the junior security. They have the following result.

Proposition 2: The total equilibrium expected revenue that the type G issuer obtains by issuing securities A and S is higher than that obtained by issuing the composite security. Thus, in equilibrium the type G firm splits its composite security into A and S. Although the total expected revenue of the type B firm is lower in the equilibrium involving securities A and S than in the equilibrium involving only the composite security, the type B firm also splits its composite security into A and S. The Nash equilibrium involving securities A and S, when augmented by the UDT's belief that a firm issuing the composite security is type B with probability one, is sequential and survives the universal divinity refinement.

Intuitively, the security splitting has two competing effects on the amount of informed demand in equilibrium. On the one hand, the informed demand increases because the expected payoff from acquiring information becomes higher on the junior security S than on the composite security, as the informed traders won't gain anything from trading the senior security A. On the other hand, the reduced liquidity demand for security S as a result of the splitting makes informed traders easier to be detected and thus discourages informed demand. Despite these two competing effects, the wealth enhancement for type G firms by splitting its security is preserved as long as there is positive informed demand in S, accompanied by sufficient liquidity demand.

This is feasible because the informed traders are able to concentrate their limited wealth in the junior security S after the splitting, instead of being confined to investing in the composite security. The "informational leveraging up" of the informed investors implies that the information acquisition cost can be compensated by a smaller difference between the true asset value and the equilibrium price level for security S than the composite security. The wealth enhancement for type G firm is qualitatively preserved as long as there is positive informed demand for S, and the splitting affects only the level of informed trading.

It is clear that a type B firm is strictly worse off to split the security rather than offer the composite security. A type B firm has to mimic the type G firm's security splitting behavior; otherwise, it is unambiguously identified as type B.

To check the robustness of the model specification above, the authors make many interesting and realistic extensions in considering factors such as market clearing with possible rationing, heterogeneous information production costs, limited short sales, and homemade splitting etc. The result in the benchmark model is qualitatively preserved under these extensions.

Now we consider the case of multiple assets and address the second research question in the paper, i.e., why bundle and re-partition individual asset's cash flows? Suppose the informed traders now get noisy signal $\tilde{\phi}_i$ for asset i. The variance of the signals (photocopied among all informed traders for each particular asset), σ, is assumed to be the same across all assets. The probability of the noise signal making type-I error is assumed to be the same as making type-II error, denoted as $\delta(\sigma) \in (0, \frac{1}{2})$. That is,

$$\Pr(\tilde{\phi}_i = G|B) = \Pr(\tilde{\phi}_i = B|G) = \delta(\sigma).$$

Since the informed traders submit zero order for perceived type B firms, we have the following expected payoff to the marginal informed trader (omitting the asset-specific subscript i)

$$\begin{aligned} V = -M + q \cdot [1 - \delta(\sigma)] \cdot \textstyle\int_0^\infty \frac{\overline{x} - P^e(\theta+l)}{P^e(\theta+l)} f(l) dl \\ -(1-q) \cdot \delta(\sigma) \cdot \textstyle\int_0^\infty \frac{P^e(\theta+l) - \underline{x}}{P^e(\theta+l)} f(l) dl. \end{aligned}$$

The equilibrium level of informed demand, θ^{**} can be determined using the zero-expected profit condition, $V(\theta^{**}|q, x, \overline{x}, M, f(l), \sigma) = 0$. The equilibrium price level can again be calculated using the Bayes' rule.

Imposing the same assumption $f'(\theta + l) < 0, \forall l \in (0, \infty)$, we can get the intuitive results $\partial V / \partial P^e > 0$ and $\partial V / \partial \theta < 0$. Finally we have the following two main results.

Proposition 5: The equilibrium measure of the set of informed traders, θ^{**}, is strictly decreasing in the idiosyncratic variance σ.

Proposition 6: The type G issuing firm's expected revenue is strictly decreasing in σ.

The first result fits well with the intuition that less precise signals reduce the marginal benefit to becoming informed. Note that $\partial \delta / \partial \sigma > 0$, i.e., the probability of type I error is increasing in signal variance. The second result implies that a type G firm would benefit from having its asset sold as part of a portfolio of type G assets. The intuition behind is that a portfolio of type G assets diversifies away the idiosyncratic noise, and the reduction in σ improves the precision of the signal received by informed investors. As a result, higher informed demand is induced and thus higher expected revenue for the issuing firm.

8.2 Diamond (1984)

Because of the post contract information asymmetry (that is, the project outcome is observable to only the entrepreneur, not the lenders), the lenders have to use costly monitoring to effectively enforce the debt contract. The costly monitoring can produce two types of inefficiency, either duplicate efforts when every lender is monitoring the same project or free rider problem when some lenders who don't monitor take advantage of others who do monitor. It seems natural to delegate the monitoring to a bank who monitors the entrepreneur on behalf of depositors so as to reduce the gross cost of monitoring. However, since the entrepreneur's payments to the bank are not observable to the depositors, when and how should the depositors delegate the monitoring task to the bank? This paper provides the determinants of delegation costs and shows that diversification among banks makes bank lending more effective than direct lending in terms of net cost incurred.

An entrepreneur with zero wealth is borrowing \$1 directly from m lenders to finance a one-period project that produces expected return higher than the risk-free rate R. Both the entrepreneur and lenders are risk-neutral and have wealth constraint – each lender has only $\$\frac{1}{m}$. The project output is a random variable $\tilde{y}$, the distribution of which is common knowledge, bounded between 0 and $\overline{y}$. The realization of output y is observable to only the entrepreneur, who sets an aggregate payment z, observable to everyone without incurring any cost and bounded between 0 and y, to the m lenders. The incentive for the entrepreneur to pay z such that $E_{\tilde{y}}(z) \geq R$ is obtained by enforcing a non-pecuniary penalty $\phi(z)$ for the entrepreneur. The non-pecuniarity of the penalty constructs a deadweight loss, as it is costly to the entrepreneur but non-beneficial to the lenders. It can be best interpreted as "bankruptcy cost" such as the grilling experience for the entrepreneur to go through the bankruptcy procedure.

The optimal contract with penalties $\phi^*(\cdot)$ solves

$$\max_{\phi(\cdot)} E_{\tilde{y}}\left[\max_{0\leq z\leq \tilde{y}} \tilde{y} - z - \phi(z)\right],$$

subject to

$$z \in \arg\max_{0\leq z\leq \tilde{y}} y - z - \phi(z),$$

and

$$E_{\tilde{y}}\left[\arg\max_{0\leq z\leq y} \tilde{y} - z - \phi(z)\right] \geq R.$$

The objective for the lenders is to set an optimal penalty $\phi(\cdot)$ such that the entrepreneur sets the payment z to maximize his own expected profit. The two constraints are the incentive compatibility conditions for the entrepreneur and lenders, respectively.

Proposition 1: The optimal contract solves the optimization above is given by $\phi^*(z) = \max(h - z, 0)$, where h is the smallest solution to

$$\Pr(\tilde{y} < h) \cdot E_{\tilde{y}}\left(\tilde{y}|y < h\right) + \Pr(\tilde{y} \geq h) \cdot h = R.$$

That is, it is a debt contract with face value h and the entrepreneur is punished to the extent to which he is no better off than telling the truth.

The face value is smallest among all debt contracts inducing truth-telling and resulting in expected return at the risk-free rate. Given the penalty function $\phi^*(\cdot)$, the incentive-compatible payment $z^* = \min(h, y)$, i.e., the entrepreneur either pays off the debt or gives up the output completely in the event of insolvency.

It is clear that any acts of purging the non-pecuniary penalty will be efficiency-improving, and costly monitoring the output $\tilde{y}$ is one such way, provided that the monitoring cost is less than the deadweight loss $\phi(\cdot)$.

When an investor can incur cost K to observe the project output $\tilde{y}$, three types of contracting scenarios may occur. The contract can be as described above, with no monitoring. A second possibility is for each of the m lenders to spend K in monitoring the output. A third choice is that the lenders can delegate the monitoring task to a bank. The least expensive of these three choices will be selected.

Since the payments to the bank are not observable to each individual lender, the m lenders on the project should provide incentive, or incur delegation cost D, to the bank to monitor and enforce the contract. The bank lending will be chosen if $K + D \leq \min\left\{E_{\tilde{y}}\left[\phi^*\left(\tilde{y}\right)\right], m \cdot K\right\}$.

Suppose that a bank with zero wealth monitors N projects defined above, receiving $g_i(y_i)$ from the i^{th} project if the bank monitors y_i. It is assumed that $\tilde{y}_i$ are independently distributed. So the bank receives total payment $G_N \equiv \sum_{i=1}^{N} g_i(y_i)$ from entrepreneurs. Let $\tilde{G}_N$ be the random variable with realization G_N, bounded between 0 and $\overline{G}_N$. The bank has to pay $N \cdot R$ in expectation, Z_N in realization, to the $N \cdot m$ depositors. It is clear that $g_i(y_i) \leq y_i$ and $Z_N \leq G_N$.

Using the same argument above, the bank has to pay the deadweight bankruptcy penalties unless $\Pr(\tilde{G}_N \geq N \cdot R) = 1$. Since each entrepreneur can pay only as much as the output received, we know $\Pr\left(\tilde{G}_N \geq N \cdot R\right) \leq \Pr\left(\sum_{i=1}^{N} y_i \geq N \cdot R\right)$. Any entrepreneur i with $\Pr(\tilde{y}_i \geq R) = 1$ could borrow directly without incurring bankruptcy penalties, hence entrepreneurs who borrow from the bank will lead the bank to incur expected bankruptcy penalties.

Let $\Phi(Z_N)$ be the bankruptcy penalties imposed on the bank. From Proposition 1, the optimal $\Phi(Z_N)$ is given by $\Phi^*(Z_N) = \max(H_N - Z_N, 0)$, where H_N is the smallest solution to

$$\Pr\left(\tilde{G}_N < H_N\right) \cdot E_{\tilde{G}_N}\left(\tilde{G}_N | G_N < H_N\right) + \Pr(\tilde{G}_N \geq H_N) \cdot H_N \geq N \cdot R.$$

The bank is not viable when $N = 1$ because it incurs as high a deadweight penalty as in absence of the bank and in addition it spends resources on monitoring. The bank as an intermediary thrives when $N \to \infty$, however, shown by Proposition 2 below.

Proposition 2: The cost of delegation per entrepreneur monitored, D_N, approaches zero as $N \to \infty$ if entrepreneurs' projects have bounded returns, distributed independently.

Proposition 2 shows that the key is the diversification in the bank's portfolio. The bank doesn't have to be monitored by the depositors in this case because the probability of payments to depositors falling short of the face value of debt contracts is very small, as a result of the bank's portfolio diversification.

Since the depositors are risk-neutral, they are not better off by investing directly in diversified projects. The diversification within the bank cannot be replaced by the depositors' diversification across banks.

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Chapter 9

Behavioral Irrationality

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9.2 Goel and Thakor (2000)

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